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mathematical models and methods

unit 22

Simultaneous differential equations

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Unit 22
Simultaneous differential equations

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1.7

Contents

Introduction

To understand the purpose of much of the mathematics in this unit, we need to look back to a group of units earlier in the course: *Unit 6* (second-order differential equations), *Unit 7* (oscillations) and *Unit 8* (damping). In *Unit 6* we studied the solution of differential equations, such as, for example,

$$\ddot{x} + 4\dot{x} + 10x = 5\cos 3t.$$

In *Units 7* and *8* we studied topics in mechanics where such second-order differential equations are used. In *Unit 7* we studied oscillations, and in particular simple harmonic motion, the 'fabulous perfect' oscillations described by equations of the form

$$\ddot{x} + \omega^2 x = 0. \tag{1}$$

In *Unit 8* we studied the model

$$\ddot{x} + 2\alpha\omega\dot{x} + \omega^2 x = 0$$

which takes account of the damping exhibited in real mechanical systems. The behaviour of such a mechanical system under an externally applied sinusoidally varying force was also studied. This may be modelled by an equation such as

$$\ddot{x} + 2\alpha\omega\dot{x} + \omega^2 x = A\cos(\nu t + \phi). \tag{2}$$

All the mechanical problems studied in *Units 7* and *8* concerned the motion of a single object in one dimension. Suppose we wish to study the motion of a mechanical system involving two (or more) objects, or of a single object whose motion cannot be described by just one position coordinate. In such cases we may arrive (say by an application of Newton's laws) at a description of the motion of such an object involving linked differential equations in more than one variable, such as

We discuss the modelling of such mechanical systems in *Unit 24*.

These linked differential equations were called 'coupled' differential equations in *Unit 15*.

$$\begin{aligned} \ddot{x} &= -\tfrac{5}{9}x + \tfrac{4}{9}y \\ \ddot{y} &= -\tfrac{4}{9}x - \tfrac{5}{9}y. \end{aligned} \tag{3}$$

In the television programme associated with this unit we will examine the motion in a plane of an object whose position coordinates (x, y) are governed by these equations. We will also see how the equations can be solved.

The substance of this unit is mathematics, although the motivation for much of its study comes from mechanics. The unit covers methods for solving certain systems of differential equations, such as (3) above, or

Although the term 'system' is usually used in this context we could equally well talk of *sets* of differential equations.

$$\begin{aligned} 2\dot{x}_1 + 3\dot{x}_2 &= x_1 - 4x_2 + e^t \\ \dot{x}_1 - 2\dot{x}_2 &= 4x_1 + x_2 + \sin t. \end{aligned} \tag{4}$$

In mechanics we are usually concerned with systems of equations, such as (3), involving second derivatives (and known as second-order systems). In this unit we will first study the simpler case of first-order systems (such as (4)) where only first derivatives occur. We will consider a method of solution for homogeneous first-order systems in Section 1, and a method for solving inhomogeneous first-order systems in Section 2. In Section 3 we study the solution of homogeneous second-order systems of a special form, similar to (3) above. Such systems may be regarded as an extension of simple harmonic motion (Equation (1)) to more than one variable. Finally, in Section 4, we look at systems of equations generalizing Equation (2) (which describes forced and damped motion) to more than one variable. We will not study the general solution of such systems, but will seek to extend to systems ideas such as 'transient' and 'steady-state solution' important in *Unit 8*. We shall also see how to compute steady-state solutions for such systems. The steady-state solution is often all that is required in mechanics.

The term 'homogeneous' is defined in Subsection 1.1, and has much the same meaning as for a single differential equation.

You have seen how systems of linear algebraic equations may conveniently be written using matrices. Systems of linear differential equations may also be expressed in matrix form. We shall find that the ideas of eigenvalue and eigenvector of a matrix, introduced in the preceding unit, are of value in solving

systems of differential equations. Thus in this unit you will need to bring together ideas from earlier units on two apparently diverse topics: differential equations and matrices.

Study guide

The sections of this unit are not of equal length. You will probably find that Sections 1 and 2 take longer to study than Sections 3 and 4.

There is no audio-tape for this unit. The television programme is part of Section 3, and the reading specifically associated with the programme is in Subsection 3.1. It is not necessary to have studied Sections 1 and 2 before watching the programme; however, you should read the pre-programme notes in Subsection 3.1 before viewing the programme. You will also find it helpful (but not essential) to have read the introduction to this unit and Subsection 1.1 before viewing.

When studying this unit you will need to be familiar with the following material from previous units:

Unit 2, Subsection 4.2 (the integrating factor technique); this is required for Subsection 2.1 of this unit.

Unit 21, particularly Sections 1 and 2 (eigenvalues and eigenvectors); this is required throughout this unit.

Unit 6, Section 4, or *Unit 8*; this is required for Section 4 of this unit, where we need:

(i) the structure of the solution of a forced oscillation equation of the form (2) above; in particular, the ideas of 'transient' and 'steady-state solution';

(ii) the calculation of steady-state solutions by the method of phasors.

Unit 2, Section 2—reviewed in *Unit 19*, Section 1—(Euler's method of numerical approximation); this is required for Subsection 2.2 of this unit.

1 First-order systems

1.1 Matrix notation and terminology

As I mentioned in the introduction, a system of linear algebraic equations may also be written in matrix notation. For example

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 5 \\ x_1 - 7x_2 + 4x_3 &= -1 \\ x_2 + x_3 &= 0 \end{aligned} \qquad \text{may be written} \qquad \begin{bmatrix} 3 & -1 & 2 \\ 1 & -7 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

or, more concisely,

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is the matrix $\begin{bmatrix} 3 & -1 & 2 \\ 1 & -7 & 4 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathbf{x}$ is the vector $[x_1 \quad x_2 \quad x_3]^T$ and $\mathbf{b}$ is the vector $[5 \quad -1 \quad 0]^T$.

Let us now consider the possibility of writing a system of *differential* equations in matrix form.

The system

$$\left.\begin{aligned} 2\dot{x}_1 + 3\dot{x}_2 &= x_1 - 4x_2 + e^t \\ \dot{x}_1 - 2\dot{x}_2 &= 4x_1 + x_2 + \sin t \end{aligned}\right\} \qquad (1)$$

can also be written

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ \sin t \end{bmatrix}.$$

To take full advantage of matrix notation we would like to write this as

$$\mathbf{A}\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{h}(t),$$

where $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$, $\mathbf{x}$ is the column vector $[x_1 \quad x_2]^T$, and $\mathbf{h}(t) = [e^t \quad \sin t]^T$. Here, the entries in the column vector $\mathbf{x}$ are *functions.* We have extended the idea of differentiation to such vectors in a natural way by writing $\dot{\mathbf{x}} = [\dot{x}_1 \quad \dot{x}_2]^T$. As far as vector algebra is concerned, column vectors of functions can be treated just like ordinary vectors.

Example 1

Suppose $\mathbf{f}(t) = [t^2 \quad 3+t \quad e^t]^T$ and $\mathbf{g}(t) = [2t^2 \quad 4 \quad e^{2t}]^T$.

Then

$$(\mathbf{f}+\mathbf{g})(t) = [3t^2 \quad 7+t \quad e^t + e^{2t}]^T \qquad \text{(addition)}$$

$$\frac{d\mathbf{f}(t)}{dt} = [2t \quad 1 \quad e^t]^T. \qquad \text{(differentiation)}$$

Exercise 1

(i) Find $\dfrac{d\mathbf{g}(t)}{dt}$, where $\mathbf{g}$ is defined in Example 1.

(ii) If $\mathbf{x} = \mathbf{a}e^{\lambda t}$, where $\mathbf{a}$ is a constant vector, find $\dot{\mathbf{x}}$.

[*Solution on p. 32*]

Using such vectors of functions, we can readily write suitable systems of differential equations in matrix form.

Example 2

Write

$$\left.\begin{aligned} \ddot{x}_1 - 3\ddot{x}_2 &= 2x_1 - x_2 \\ 2\ddot{x}_1 + 4\ddot{x}_2 &= x_1 + 3x_2 + \sin 4t \end{aligned}\right\} \qquad (2)$$

in matrix form.

Solution

In matrix form System (2) becomes

$$\mathbf{A}\ddot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{h}(t)$$

where $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $\mathbf{x} = [x_1 \quad x_2]^T$ and $\mathbf{h}(t) = [0 \quad \sin 4t]^T$.

The matrix notation need not be restricted to systems of two equations in two unknowns, nor to matrices with constant entries.

Example 3

Write the following system of equations in matrix form.

$$\left.\begin{aligned} t\dot{x}_1 - x_3 &= \sin t \\ \dot{x}_2 + \dot{x}_3 + tx_1 &= 0 \\ (\sin t)\dot{x}_2 - \dot{x}_1 &= 2x_2 \end{aligned}\right\} \qquad (3)$$

Solution

First write the system as

$$\begin{aligned} &t\dot{x}_1 & & & -x_3 &= \sin t \\ & & \dot{x}_2 + \dot{x}_3 + tx_1 & & &= 0 \\ &-\dot{x}_1 + (\sin t)\dot{x}_2 & & -2x_2 & &= 0. \end{aligned}$$

Then it is easily seen to be equivalent to

$$\mathbf{A}(t)\dot{\mathbf{x}} + \mathbf{B}(t)\mathbf{x} = \mathbf{h}(t)$$

where $\mathbf{A}(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 1 \\ -1 & \sin t & 0 \end{bmatrix}$, $\mathbf{B}(t) = \begin{bmatrix} 0 & 0 & -1 \\ t & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$, $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$

and $\mathbf{h}(t) = [\sin t \quad 0 \quad 0]^T$.

However, not every system of differential equations can be written in matix form, in the way we have in these examples. For example the system

$$\left.\begin{aligned} \dot{x}_1^2 + \dot{x}_2^2 &= x_1 + x_2 \\ \dot{x}_1\dot{x}_2 &= x_1 - x_2 \end{aligned}\right\} \qquad (4)$$

could not be.

In this unit we shall be concerned almost exclusively with systems of the form

$$\mathbf{A}_1(t)\dot{\mathbf{x}} + \mathbf{A}_2(t)\mathbf{x} = \mathbf{h}(t) \qquad (5)$$

or of the form

$$\mathbf{A}_1(t)\ddot{\mathbf{x}} + \mathbf{A}_2(t)\dot{\mathbf{x}} + \mathbf{A}_3(t)\mathbf{x} = \mathbf{h}(t) \qquad (6)$$

(where $\mathbf{A}_1(t)$ is not the zero matrix in either case). Such systems are said to be **linear**. Equation (5) is a linear **first-order** system (it contains no derivative higher than first-order). Equation (6) is a linear **second-order** system. These definitions of 'linear' and 'order' are extensions of those for a single differential equation. We shall usually be concerned with **constant-coefficient** systems, in which all the entries in the matrices $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$ and $\mathbf{A}_3(t)$ are constants (though $\mathbf{h}(t)$ may depend on t). If $\mathbf{h}(t) = \mathbf{0}$ in (5) or (6), then we have a **homogeneous** linear system.

See the introduction to *Unit 6*.

Exercise 2

Which of the systems (1)–(4) in the text above are linear? Of those that are linear, say which (if any) are homogeneous, and which (if any) are constant-coefficient, and give the order in each case.

[*Solution on p. 32*]

Exercise 3

(i) Write the system

$$\begin{aligned} \dot{x}_1 + 2x_1 &= 3\dot{x}_2 - x_2 \\ \dot{x}_3 &= x_1 + x_2 + x_3 \\ \dot{x}_2 &= \dot{x}_1 + \dot{x}_3 \end{aligned}$$

in matrix form.

(ii) Is this system constant-coefficient? Is it homogeneous? What is its order?

[*Solution on p. 32*]

In the remainder of this section we are concerned with methods of solution of linear, constant-coefficient, first-order systems of differential equations. In each case the first step in the method of solution is to write the system in normal form as defined below.

> A linear constant-coefficient first-order system is in **normal form** when it is written in the form
>
> $$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{k}(t).$$

The general constant-coefficient linear first-order system is

$$\mathbf{A}_1\dot{\mathbf{x}} + \mathbf{A}_2\mathbf{x} = \mathbf{h}(t).$$

This can be written in normal form so long as the matrix $\mathbf{A}_1$ is non-singular. For then we have

$$\dot{\mathbf{x}} = -\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{x} + \mathbf{A}_1^{-1}\mathbf{h}(t),$$

which is in normal form.

Exercise 4

Where possible, write each of the systems of differential equations below in normal form. If it is not possible, explain why.

(i) $\dot{x}_1 + 2\dot{x}_2 + 5x_1 = t$
$2\dot{x}_1 - \dot{x}_2 + 10x_2 = 2$

(ii) $\dot{x}_1 + \dot{x}_2 + 5x_1 = t$
$2\dot{x}_1 + 2\dot{x}_2 + x_1 - x_2 = 0$

(iii) $\dot{x}_1^2 + \dot{x}_2 = 5t^2$
$\dot{x}_1 + \dot{x}_2^2 = 0$

(iv) $\dot{x}_1 + \dot{x}_2 = tx_1 - x_2$
$\dot{x}_1 - \dot{x}_2 = x_1 + 2x_2$

[*Solution on p. 32*]

1.2 The solution of constant-coefficient homogeneous systems

We now consider a method of solution for constant-coefficient, first-order, homogeneous systems.

Method of solution

Let us consider a constant-coefficient, first-order, homogeneous system in normal form:

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}. \tag{7}$$

We can derive quite a simple formula for the solution of (7), based on eigenvalues and eigenvectors. Let us first see where the formula comes from. To do this, we look for solutions of (7) which have the form

$$\mathbf{x} = \mathbf{a}e^{\lambda t} \tag{8}$$

where $\mathbf{a}$ is a non-zero constant vector (and λ is a constant scalar).

If $\mathbf{a} = \mathbf{0}$ we get $\mathbf{x} = \mathbf{0}$, which is a solution of (7), but not a very interesting one.

If $\mathbf{x} = \mathbf{a}e^{\lambda t}$, as in (8), then

$$\dot{\mathbf{x}} = \lambda\mathbf{a}e^{\lambda t} \qquad \text{(see Exercise 1(ii))}.$$

Thus (8) is a solution of (7), so long as

$$\lambda\mathbf{a}e^{\lambda t} = \mathbf{B}\mathbf{a}e^{\lambda t}.$$

That is, so long as

$$\lambda\mathbf{a} = \mathbf{B}\mathbf{a} \qquad (\text{where } \mathbf{a} \neq \mathbf{0}). \tag{9}$$

But this equation is just the definition of the eigenvalues and eigenvectors of the matrix $\mathbf{B}$. So we see that $\mathbf{x} = \mathbf{a}e^{\lambda t}$ is a non-zero solution of $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ so long as λ is an eigenvalue of the matrix $\mathbf{B}$ and $\mathbf{a}$ is a corresponding eigenvector.

Now the matrix $\mathbf{B}$ will (usually) have more than one eigenvalue, so $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ will (usually) have several solutions of the form $\mathbf{x} = \mathbf{a}e^{\lambda t}$. When we have a homogeneous linear system of differential equations it is quite easily proved that any linear combination of solutions is again a solution. So if λ_1 and λ_2 are eigenvalues of $\mathbf{B}$, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are corresponding eigenvectors, then

$$\mathbf{x} = C_1\mathbf{a}_1e^{\lambda_1 t} + C_2\mathbf{a}_2e^{\lambda_2 t}$$

is also a solution of (7) (where C_1 and C_2 are arbitrary constants).

We have found a number of solutions of (7). The only question that remains is whether we can construct the *general* solution of (7) by taking linear combinations of exponentials in this way. Unfortunately the answer is 'not always'. The matrix $\mathbf{B}$ must satisfy the conditions given in Theorem 1 below. Fortunately, many of the matrices which occur in practice do satisfy the conditions of the theorem.

Theorem 1

Let $\mathbf{B}$ be a square, $n \times n$, matrix. Suppose that $\mathbf{B}$ has n linearly independent eigenvectors, say $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, corresponding to the (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the general solution of the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$$

is

$$\mathbf{x} = C_1\mathbf{a}_1e^{\lambda_1 t} + C_2\mathbf{a}_2e^{\lambda_2 t} + \ldots + C_n\mathbf{a}_ne^{\lambda_n t},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

You may notice a similarity to the general solution of a homogeneous second-order differential equation, in *Unit 6*.

Theorem 1, then, provides a procedure for finding the general solution of a homogeneous linear first-order system, so long as the system satisfies the conditions of the theorem.

Procedure 1.2: To solve a linear, constant-coefficient, homogeneous, first-order system

For this procedure to work we need to be able to express the system in the normal form

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$$

where $\mathbf{B}$ is an $n \times n$ matrix with n linearly independent eigenvectors.

1. Express the system of equations in the normal form $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$.
2. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{B}$ and a corresponding set of linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.
3. Write down the general solution of $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ in the form

$$\mathbf{x} = C_1\mathbf{a}_1 e^{\lambda_1 t} + C_2\mathbf{a}_2 e^{\lambda_2 t} + \ldots + C_n\mathbf{a}_n e^{\lambda_n t}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example 4

Find the general solution of

$$\left.\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= 3x_1 - x_2. \end{aligned}\right\}$$

Solution

The system is already in normal form $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ where

Step 1

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

We must find the eigenvalues and eigenvectors of $\mathbf{B}$.

Step 2

The eigenvalues are found by solving the characteristic equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. Now

$$\begin{aligned} \det(\mathbf{B} - \lambda\mathbf{I}) &= \begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \\ &= (1-\lambda)(-1-\lambda) - 3 \\ &= \lambda^2 - 4. \end{aligned}$$

So the eigenvalues are 2 and -2.

To find corresponding eigenvectors we solve

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = 2$: we obtain the equations

$$\begin{aligned} -u + v &= 0 \\ 3u - 3v &= 0 \end{aligned}$$

which have solutions of the form $u = k$, $v = k$. So an eigenvector corresponding to the eigenvalue 2 is $[1 \quad 1]^T$.

The general eigenvector is $[k \quad k]^T$, but we only need one eigenvector here.

Case $\lambda = -2$: we obtain the equations

$$3u + v = 0 \qquad \text{(twice)}$$

so an eigenvector corresponding to the eigenvalue -2 is $[1 \quad -3]^T$.

Since the conditions of Procedure 1.2 are satisfied (**B** is 2×2 with 2 linearly independent eigenvectors) we can write down the general solution:

$$\mathbf{x} = C_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t}.$$

That is

$$x_1 = C_1e^{2t} + C_2e^{-2t}$$
$$x_2 = C_1e^{2t} - 3C_2e^{-2t}$$

where C_1 and C_2 are arbitrary constants.

The choice of eigenvectors is not unique. Other choices lead to different looking, but equivalent, forms of the general solution.

It is sensible to check solutions to differential equations. This is readily done here. With x_1 and x_2 as above, we have

$$\dot{x}_1 = 2C_1e^{2t} - 2C_2e^{-2t},$$

while

$$x_1 + x_2 = 2C_1e^{2t} - 2C_2e^{-2t}.$$

So $\dot{x}_1 = x_1 + x_2$, as required.

Similarly

$$\dot{x}_2 = 2C_1e^{2t} + 6C_2e^{-2t}$$
$$= 3x_1 - x_2.$$

Exercise 5

Find the general solution of

$$\dot{x}_1 = 2x_1 + 3x_2$$
$$\dot{x}_2 = 2x_1 + x_2$$

[*Solution on p. 32*]

An example with a repeated eigenvalue

For Procedure 1.2 to work the matrix **B** must have n linearly independent eigenvectors. It does not matter if there are less than n distinct eigenvalues provided there are enough linearly independent eigenvectors corresponding to each eigenvalue to give a total of n linearly independent eigenvectors. Let us see an example of this.

See *Unit 21*, Subsection 1.4, for a discussion of eigenvectors corresponding to repeated eigenvalues

Example 5

Find the general solution of the system

$$\dot{x}_1 = 5x_1 + 3x_3$$
$$\dot{x}_2 = 3x_1 + 2x_2 + 3x_3$$
$$\dot{x}_3 = -6x_1 - 4x_3.$$

Solution

The equations are in normal form, as required. We have $\dot{\mathbf{x}} = \mathbf{Bx}$ with

Step 1

$$\mathbf{B} = \begin{bmatrix}5 & 0 & 3\\3 & 2 & 3\\-6 & 0 & -4\end{bmatrix}.$$

We need to find the eigenvalues and eigenvectors of **B**.

Step 2

The eigenvalues are found by solving the characteristic equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. We have

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix}5-\lambda & 0 & 3\\3 & 2-\lambda & 3\\-6 & 0 & -4-\lambda\end{vmatrix} = -\begin{vmatrix}0 & 5-\lambda & 3\\2-\lambda & 3 & 3\\0 & -6 & -4-\lambda\end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 3 & 3 \\ 0 & 5-\lambda & 3 \\ 0 & -6 & -4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\big((5-\lambda)(-4-\lambda)+18\big)$$

$$= (2-\lambda)(\lambda^2-\lambda-2)$$

$$= -(\lambda-2)^2(\lambda+1).$$

So the eigenvalues are 2 (a repeated eigenvalue) and -1.

To find corresponding eigenvectors we solve

$$\begin{bmatrix} 5-\lambda & 0 & 3 \\ 3 & 2-\lambda & 3 \\ -6 & 0 & -4-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{0}$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = -1$ gives

$$6u + 3w = 0$$

$$3u + 3v + 3w = 0$$

$$-6u - 3w = 0.$$

Putting $u = k$, the first and third equations give $w = -2k$. The second equation then gives $v = k$. So an eigenvector corresponding to the eigenvalue -1 is $[1 \quad 1 \quad -2]^T$.

Case $\lambda = 2$ gives

$$3u + 3w = 0$$

$$3u + 3w = 0$$

$$-6u - 6w = 0.$$

So, putting $w = k$ we have $u = -k$. Now v can take any value l say. Thus, any vector of the form $[-k \quad l \quad k]^T$ is an eigenvector. In particular $[0 \quad 1 \quad 0]^T$ and $[-1 \quad 0 \quad 1]^T$ are linearly independent eigenvectors corresponding to the eigenvalue 2.

In this case, even though there are only two distinct eigenvalues, there are sufficient linearly independent eigenvectors (three) for the conditions of Procedure 1.2 to be satisfied. Thus the general solution of the given system is

$$\mathbf{x} = C_1\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{2t} + C_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{2t}.$$

In terms of x_1, x_2 and x_3, the solution is

$$x_1 = C_1e^{-t} - C_3e^{2t}$$

$$x_2 = C_1e^{-t} + C_2e^{2t}$$

$$x_3 = -2C_1e^{-t} + C_3e^{2t}.$$

Although the above example requires considerable calculation, there is little that is new in it. Most of it is a use of matrix methods you have met in earlier units.

An example with complex eigenvectors

In attempting to apply Theorem 1, it is quite possible that we may find that some or all of the eigenvalues and eigenvectors of the matrix **B** are complex.

Example 6

Find the general solution of

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1\end{aligned}\right\} \tag{10}$$

Solution

We have $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ where Step 1

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

To find the eigenvalues of $\mathbf{B}$, look at the characteristic equation: Step 2

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0,$$

i.e. $\lambda^2 + 1 = 0$.

So the eigenvalues are i and $-i$.

To find corresponding eigenvectors we solve

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = i$ gives

$$-iu + v = 0$$

$$-u - iv = 0.$$

The first equation is i times the second and so both give $v = iu$. Thus an eigenvector corresponding to the eigenvalue i is $[1 \quad i]^T$.

Case $\lambda = -i$ gives

$$iu + v = 0$$

$$-u + iv = 0.$$

The second equation is i times the first and so both give $u = iv$. Thus an eigenvector corresponding to the eigenvalue $-i$ is $[1 \quad -i]^T$.

So by Theorem 1, the general solution is given by linear combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} = \begin{bmatrix} e^{it} \\ ie^{it} \end{bmatrix} = \begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

and

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-it} = \begin{bmatrix} e^{-it} \\ -ie^{-it} \end{bmatrix} = \begin{bmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} - i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

Since both $\mathbf{x}_1$ and $\mathbf{x}_2$ are linear combinations of $[\cos t \quad -\sin t]^T = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ and $[\sin t \quad \cos t]^T = \dfrac{1}{2i}(\mathbf{x}_1 - \mathbf{x}_2)$, the general solution can equally well be written

$$\mathbf{x} = C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

For our purposes this is a much better way of writing the general solution because we are only interested in real solutions.

The approach in the above example works because $\mathbf{x}_1$ and $\mathbf{x}_2$ form a complex conjugate pair with real part $[\cos t \quad -\sin t]^T$ and imaginary part $[\sin t \quad \cos t]^T$. It turns out that systems for which $\mathbf{B}$ has complex eigenvalues

can always be treated in this way provided **B** is real (i.e. all the elements of **B** are real). The justification for this is provided by the following theorem.

> **Theorem 2**
>
> If the conditions of Theorem 1 are satisfied, and in addition **B** is real, then the terms in the general solution
>
> $$\mathbf{x} = C_1\mathbf{a}_1e^{\lambda_1 t} + C_2\mathbf{a}_2e^{\lambda_2 t} + \ldots + C_n\mathbf{a}_ne^{\lambda_n t}$$
>
> are either real or occur in conjugate pairs $\mathbf{a}e^{\lambda t}$ and $\bar{\mathbf{a}}e^{\bar{\lambda}t}$.

$\bar{\mathbf{a}}$ is the vector obtained by taking the complex conjugate of each element of **a**.

In Example 6 we used Procedure 1.2 to find the general solution in the form

$$\mathbf{x} = C_1\mathbf{a}e^{\lambda t} + C_2\bar{\mathbf{a}}e^{\bar{\lambda}t}$$

where $\mathbf{a} = [1 \quad i]^T$ and $\lambda = i$. We then used the fact that the terms in this solution are conjugate to write the solution in the real form

$$\mathbf{x} = C_1\mathrm{Re}(\mathbf{a}e^{\lambda t}) + C_2\mathrm{Im}(\mathbf{a}e^{\lambda t})$$

where $\mathrm{Re}(\mathbf{a}e^{\lambda t}) = [\cos t \quad -\sin t]^T$ is the real part of $\mathbf{a}e^{\lambda t}$ and $\mathrm{Im}(\mathbf{a}e^{\lambda t}) = [\sin t \quad \cos t]^T$ is the imaginary part of $\mathbf{a}e^{\lambda t}$.

More generally, Theorem 2 states that if the general solution in Procedure 1.2 contains complex terms then they occur in conjugate pairs. It is therefore possible to obtain a real form for the general solution by using the following procedure.

> **Procedure 1.2(a): Applicable when the matrix B in Procedure 1.2 is real but has some complex eigenvalues**
>
> Suppose that **a** is a complex eigenvector corresponding to the eigenvalue λ. Then replace the terms $\mathbf{a}e^{\lambda t}$ and $\bar{\mathbf{a}}e^{\bar{\lambda}t}$ appearing in Procedure 1.2 by $\mathrm{Re}(\mathbf{a}e^{\lambda t})$ and $\mathrm{Im}(\mathbf{a}e^{\lambda t})$.

The method of solution of $\dot{\mathbf{x}} = \mathbf{Bx}$ given in this section only works if **B** has enough linearly independent eigenvectors. In applications the matrix **B** often turns out to be symmetric. When this happens Procedure 1.2 is always applicable, for symmetric matrices always have a full set of linearly independent eigenvectors.

The eigenvectors of symmetric matrices were discussed briefly at the end of *Unit 21*, Section 1.

Exercise 6

Find the general solution of each of the systems below.

(i) $\dot{x}_1 = 5x_1 + 4x_2$
$\dot{x}_2 = -x_1$

(ii) $\dot{x}_1 = 5x_1 - 6x_2 - 6x_3$
$\dot{x}_2 = -x_1 + 4x_2 + 2x_3$
$\dot{x}_3 = 3x_1 - 6x_2 - 4x_3$
(Hint: $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2$).

(iii) $\dot{x}_1 = -2x_1 + 2x_2$
$\dot{x}_2 = -x_1$

[*Solution on p. 32*]

Summary of Section 1

We can use matrix notation to represent suitable systems of differential equations. A linear first-order system is in **normal form** if it is written

$$\dot{\mathbf{x}} = \mathbf{Bx} + \mathbf{k}.$$

We can use Procedure 1.2 to solve a homogeneous, constant-coefficient system in the normal form, $\dot{\mathbf{x}} = \mathbf{Bx}$, so long as **B** has enough linearly independent eigenvectors. We saw also in Procedure 1.2(a) how to write the solution in an explicitly real form, when some of the eigenvalues and eigenvectors of **B** are complex.

2 Further methods for linear first-order systems

2.1 Constant-coefficient inhomogeneous systems

Suppose we wish to solve an inhomogeneous system $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{h}(t)$, such as

$$\left.\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 + t \\ \dot{x}_2 &= 2x_1 + x_2 + 4t. \end{aligned}\right\} \qquad (1)$$

So long as $\mathbf{B}$ has 'enough' eigenvectors (as in Theorem 1 of Section 1), it is possible to transform such a system into a much simpler system. We shall see shortly that by setting $x_1 = 3y_1 + y_2$ and $x_2 = 2y_1 - y_2$, System (1) becomes

$$\left.\begin{aligned} \dot{y}_1 &= 4y_1 + t \\ \dot{y}_2 &= -y_2 - 2t. \end{aligned}\right\} \qquad (2)$$

Each equation in (2) can be solved by the integrating factor method of *Unit 2*, since each equation now involves only one unknown variable. We can then obtain the solution of (1) from that for (2).

Before describing the general procedure, let us see an example of how it works. I shall make a note of the steps involved in the margin.

Example 1

Solve System (1) above.

Method

The system may be written

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{h}(t) \qquad (3)$$

where $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{h}(t) = [t \quad 4t]^T$.

The first step is to find the eigenvalues and eigenvectors of the matrix $\mathbf{B}$. In fact we have already done this in the solution to Exercise 5 of Section 1 where we found that:

$[3 \quad 2]^T$ is an eigenvector corresponding to the eigenvalue 4;

$[1 \quad -1]^T$ is an eigenvector corresponding to the eigenvalue -1.

The next step is to form a matrix $\mathbf{P}$, whose columns are the eigenvectors. So here,

Are there enough eigenvectors? If so, form $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is diagonal.

$$\mathbf{P} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}.$$

We then know (by Theorem 5 of *Unit 21*, Section 3) that $\mathbf{P}$ is non-singular and that

$$\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

That is, $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is a diagonal matrix with diagonal entries equal to the eigenvalues. We now substitute $\mathbf{x} = \mathbf{P}\mathbf{y}$ in (3), where $\mathbf{y}$ is a new vector of variables. This gives

Put $\mathbf{x} = \mathbf{P}\mathbf{y}$

$$\mathbf{P}\dot{\mathbf{y}} = \mathbf{B}\mathbf{P}\mathbf{y} + \mathbf{h}(t).$$

(Here we have used the fact that since all the entries in $\mathbf{P}$ are constant, differentiating $\mathbf{P}\mathbf{y}$ with respect to t gives $\mathbf{P}\dot{\mathbf{y}}$.)

Since $\mathbf{P}$ is non-singular we can multiply by $\mathbf{P}^{-1}$ to obtain

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\mathbf{y} + \mathbf{P}^{-1}\mathbf{h}(t).$$

Calculate $\mathbf{P}^{-1}$.

Because $\mathbf{P}$ is a 2×2 matrix we can easily calculate its inverse:

$$\mathbf{P}^{-1} = -\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix}.$$

Thus, in terms of the $\mathbf{y}$ variables, System (1) becomes

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} t \\ 4t \end{bmatrix}.$$

That is,

$$\dot{y}_1 = 4y_1 + t$$

$$\dot{y}_2 = -y_2 - 2t.$$

We now have a system that we can solve, because each equation can be solved by the integrating factor method of *Unit 2*. For example, the first equation can be written

Solve the resulting equations in $\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\mathbf{y} + \mathbf{P}^{-1}\mathbf{h}$ individually, by the integrating factor method.

$$\dot{y}_1 - 4y_1 = t.$$

Multiplying both sides by the integrating factor $e^{-\int 4dt} = e^{-4t}$ gives

$$e^{-4t}(\dot{y}_1 - 4y_1) = te^{-4t}$$

that is

$$\frac{d}{dt}(y_1 e^{-4t}) = te^{-4t}.$$

Integrating both sides we obtain

$$y_1 e^{-4t} = \int te^{-4t} dt + C_1$$

$$= (-\tfrac{1}{4}t - \tfrac{1}{16})e^{-4t} + C_1 \qquad \text{(integrating by parts)},$$

thus

$$y_1 = C_1 e^{4t} - \tfrac{1}{4}t - \tfrac{1}{16}.$$

The second equation may be solved similarly to give (omitting the details)

$$y_2 = C_2 e^{-t} - 2t + 2.$$

Now the solution of the original system is $\mathbf{x} = \mathbf{P}\mathbf{y}$. That is,

Now find $\mathbf{x}$, from $\mathbf{x} = \mathbf{P}\mathbf{y}$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} C_1 e^{4t} - \frac{1}{4}t - \frac{1}{16} \\ C_2 e^{-t} - 2t + 2 \end{bmatrix}.$$

Thus

$$x_1 = 3C_1 e^{4t} + C_2 e^{-t} - \tfrac{11}{4}t + \tfrac{29}{16}$$

$$x_2 = 2C_1 e^{4t} - C_2 e^{-t} + \tfrac{3}{2}t - \tfrac{17}{8}.$$

So we have solved the original system (1).

(The result of such a calculation should of course be checked, though I will not give details here.)

Check

We could have used a method similar to the above for the homogeneous systems in Subsection 1.2, but there the above method would have been unnecessarily complex. Here we need to make the transformation $\mathbf{x} = \mathbf{P}\mathbf{y}$ to get the system into a 'diagonal' form, that can be solved equation-by-equation using the integrating factor method.

The procedure for inhomogeneous systems is set out below.

Procedure 2.1: To solve a linear, constant-coefficient, inhomogeneous, first-order system

For this procedure to work we need to be able to express the system in the normal form

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{h}(t) \tag{3}$$

where $\mathbf{B}$ is an $n \times n$ matrix with n linearly independent eigenvectors.

1. (i) Find n linearly independent eigenvectors of $\mathbf{B}$;

   (ii) Form a matrix $\mathbf{P}$ whose columns are these eigenvectors;

   (iii) Find $\mathbf{P}^{-1}$.

2. Put

   $$\mathbf{x} = \mathbf{P}\mathbf{y}$$

   in (3), to obtain

   $$\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\mathbf{y} + \mathbf{P}^{-1}\mathbf{h}(t), \tag{4}$$

   where $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is diagonal.

3. Find $\mathbf{y}$ by solving the equations in (4) individually, using the integrating factor method.

4. Transform the solution for $\mathbf{y}$ back to a solution for $\mathbf{x}$, using $\mathbf{x} = \mathbf{P}\mathbf{y}$.

Exercise 1

Solve the system

$$\dot{x}_1 = 2x_1 + 3x_2 + e^{2t}$$

$$\dot{x}_2 = 2x_1 + x_2 + 4e^{2t}.$$

[*Solution on p. 34*]

Exercise 2

Solve the system

$$\dot{x}_1 = -2x_1 + 10x_2 + 48$$

$$\dot{x}_2 = 10x_1 - 2x_2 + 91e^{t}.$$

[*Solution on p. 34*]

2.2 Numerical methods

We saw in *Units 2* and *6* that differential equations cannot always be solved by analytical methods. In such a situation we may resort to obtaining numerical approximations to the solution. Techniques for doing this were discussed extensively in *Unit 19*.

We are now faced with a similar situation for systems of differential equations. The analytical techniques described in this unit are only applicable to certain special types of systems and so the need to have numerical techniques for systems of differential equations is as pressing as it was for single differential equations.

Although, for reasons of space, I cannot give a comprehensive treatment of the topic, it is useful to indicate here how we can go about constructing numerical schemes to generate approximate solutions to first-order systems of differential equations. We will look at a numerical method based on Euler's method for a

Euler's method was discussed in *Units 2* and *19*.

single differential equation. Recall that in Euler's method for a single differential equation we approximate the solution of the initial condition problem

$$\frac{dx}{dt} = m(x, t) \qquad x = x_0 \text{ when } t = t_0$$

by starting with $X_0 = x_0$ and using the recurrence relation

$$X_{r+1} = X_r + hm(X_r, t_r)$$

to generate approximations X_r to the true solution $x(t_r)$ at $t_r = t_0 + rh$.

We can easily extend Euler's method to a system of two equations of the form

$$\dot{x}_1 = m_1(x_1, x_2, t)$$

$$\dot{x}_2 = m_2(x_1, x_2, t).$$

We do this by using Euler's method to generate approximate solutions to each of these equations. Looking at the first, Euler's method gives an approximation $X_{1,r}$ to $x_1(t_r)$ where

$$X_{1,r+1} = X_{1,r} + hm_1(X_{1,r}, X_{2,r}, t_r).$$

Here $X_{2,r}$ is the approximation to $x_2(t_r)$ where (applying Euler's method to the second equation)

$$X_{2,r+1} = X_{2,r} + hm_2(X_{1,r}, X_{2,r}, t_r).$$

This may seem a little complicated but in practice it is quite straightforward. Let us look at an example.

Example 2

Write down the recurrence relations corresponding to an application of Euler's method to the system

$$\dot{x}_1 = tx_1 - t^2x_2$$

$$\dot{x}_2 = 3x_1 - \cos t,$$

where $x_1 = 1$, $x_2 = -3$ at $t = 0$.

Solution

Applying Euler's method to each equation in turn gives

$$\left.\begin{aligned} X_{1,r+1} &= X_{1,r} + h(t_r X_{1,r} - t_r^2 X_{2,r}) \\ X_{2,r+1} &= X_{2,r} + h(3X_{1,r} - \cos t_r). \end{aligned}\right\} \qquad (5)$$

We have also that $X_{1,0} = 1$, $X_{2,0} = -3$, and that $t_r = rh$.

The recurrence relations (5) obtained in the Example can be used to generate approximations to the solution of the given system, step-by-step. Starting from the known values of $X_{1,0}$ and $X_{2,0}$, we use (5) to calculate $X_{1,1}$ and $X_{2,1}$. We then use (5) again to calculate $X_{1,2}$ and $X_{2,2}$; and so on. The results obtained depend on the step-length h chosen, and will usually be more accurate the smaller h is.

Example 3

Use the recurrence relations (5) to calculate $X_{1,2}$ and $X_{2,2}$ with $h = 0.1$. What values of the solutions do these approximate?

Solution

Putting $r = 0$ in (5), we obtain

$$X_{1,1} = X_{1,0} + 0.1\ (t_0 X_{1,0} - t_0^2 X_{2,0})$$

$$X_{2,1} = X_{2,0} + 0.1\ (3X_{1,0} - \cos t_0).$$

Substituting $X_{1,0} = 1$, $X_{2,0} = -3$, and $t_0 = 0$, we obtain

$$X_{1,1} = 1$$

$$X_{2,1} = -3 + 0.1(3 - 1) = -2.8.$$

Putting $r = 1$ in (5) gives

$$X_{1,2} = X_{1,1} + 0.1\,(0.1X_{1,1} - 0.01X_{2,1})$$
$$= 1 + 0.1\,(0.1 + 0.028)$$
$$= 1.0128.$$
$$X_{2,2} = -2.8 + 0.1\,(3 - \cos 0.1)$$
$$= -2.5995.$$

Since we are working with step length $h = 0.1$, $X_{1,2}$ is the calculated approximation to $x_1(0.2)$, and $X_{2,2}$ the approximation to $x_2(0.2)$.

Calculations such as this are time-consuming and would normally be performed on a computer, so I shall do no more by hand here.

It is a straightforward matter to extend the above method to systems containing more than two equations. We can consider systems of equations which can be written in the form

$$\dot{x}_1 = m_1(x_1, x_2, \ldots, x_n, t)$$
$$\dot{x}_2 = m_2(x_1, x_2, \ldots, x_n, t)$$
$$\vdots$$
$$\dot{x}_n = m_n(x_1, x_2, \ldots, x_n, t).$$

If we introduce the vector notation $\mathbf{x} = [x_1 \quad x_2 \quad \ldots \quad x_n]^T$ these equations can be written more concisely in the form

$$\dot{\mathbf{x}} = \mathbf{m}(\mathbf{x}, t).$$

The procedure for applying Euler's method to such systems is given below.

Procedure 2.2: Euler's method for systems of equations

This procedure applies to systems of the form

$$\dot{\mathbf{x}} = \mathbf{m}(\mathbf{x}, t)$$

with initial condition $\mathbf{x} = \mathbf{x}_0$ at $t = t_0$.

1. Replace each equation in the system by a recurrence relation, using Euler's method. Thus the equation

   $$\dot{x}_i = m_i(\mathbf{x}, t)$$

   is replaced by

   $$X_{i,r+1} = X_{i,r} + hm_i(\mathbf{X}_r, t_r). \qquad (i = 1, 2, \ldots, n) \qquad (8)$$

   Here $\mathbf{X}_r = [X_{1,r} \quad X_{2,r} \quad \ldots \quad X_{n,r}]^T$ is the approximation to $\mathbf{x}(t_r)$ where $t_r = t_0 + rh$.

2. (i) Set $\mathbf{X}_0 = \mathbf{x}_0$.

   (ii) Choose a step length h.

   (iii) Use the recurrence relations (8) to calculate $\mathbf{X}_1$, then use (8) again to calculate $\mathbf{X}_2$, and so on.

Exercise 3

(i) Write down, in a form suitable for generating solutions, the recurrence relations corresponding to an application of Euler's method to

$$\dot{x}_1 = 3tx_2 + 4$$
$$\dot{x}_2 = tx_1 - x_2 - e^t,$$

where $x_1 = 5$, $x_2 = 2$ at $t = 0$.

(ii) Use the recurrence relations you obtain, with step-length 0.1, to calculate approximations to $x_1(0.2)$ and $x_2(0.2)$.

[*Solution on p. 34*]

We have seen in previous units that care must be taken in the use of numerical methods. There is no time here for a discussion of the important topic of stability, which is essential to a proper understanding of numerical schemes such as those constructed above. Approaches other than those based on Euler's method are possible; in fact we could develop methods for first-order systems based on any of the methods for a single first-order equation described in *Unit 19*, but, again, I will not pursue this topic here.

For a discussion of stability see *Unit 19.*

Summary of Section 2

To solve an inhomogeneous constant-coefficient system of differential equations $\dot{\mathbf{x}} = \mathbf{Bx} + \mathbf{h}(t)$ where $\mathbf{B}$ is an $n \times n$ matrix with n linearly independent eigenvectors, we use Procedure 2.1.

A method of constructing recurrence relations to generate approximate solutions to systems of differential equations was described. This is based on Euler's method, and consists of applying the approximation specified by Euler's method to each equation in the system in turn.

3 Second-order homogeneous systems of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$: simple harmonic motion

We now turn our attention to systems of differential equations which involve second derivatives. We shall be concerned solely with linear, constant-coefficient, second-order systems. Such systems arise frequently in modelling, especially in mechanics. A particularly important class of second-order systems are those that can be written in the form $\ddot{\mathbf{x}} = \mathbf{Bx}$, and we study these in this section.

Subsection 3.1 summarizes material covered in the television programme 'Two-way motion'. The programme looks at the solution of a particular system of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$ and illustrates the sort of real situation in which such a system of differential equations may arise. In Subsection 3.2 we look at a procedure for solving systems of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$.

3.1 'Two-way motion' (Television Subsection)

Pre-programme notes

In the programme we study the solution of the pair of differential equations

$$\left.\begin{aligned} \ddot{x} &= -\tfrac{5}{9}x + \tfrac{4}{9}y \\ \ddot{y} &= \tfrac{4}{9}x - \tfrac{5}{9}y. \end{aligned}\right\} \qquad (1)$$

This pair of differential equations provides an approximate description of the motion of a ball-bearing in a suitably shaped bowl (where x and y are the horizontal position coordinates of the ball-bearing). The programme consists of three parts:

1. The actual motion of the ball-bearing in such a bowl;
2. A computer animation of the precise motion implied by Equations (1);
3. The mathematical solution of Equations (1).

(The computer animation enables us to see how accurately Equations (1) describe the actual motion of the ball-bearing. For various reasons the actual motion is only rather crudely approximated by Equations (1). For example the equations take no account of various factors tending to damp the motion.)

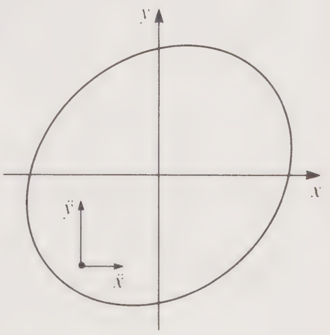


Figure 1. We consider the motion of a ball-bearing in a bowl. The two horizontal components of acceleration, $\ddot{x}$ and $\ddot{y}$, are specified (approximately) by the Equations (1).

To solve the equations mathematically it is helpful to use vector notation. If we write $\mathbf{r}$ for the vector of variables $[x \quad y]^T$ and $\mathbf{A}$ for the matrix

$$\begin{bmatrix} -\frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix},$$

You will find this notation discussed further in Subsection 1.1.

then we see that the pair of differential equations (1) can be written

$$\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}.$$

This equation is similar in form to the single differential equation

$$\ddot{y} = ky.$$

For $k < 0$, this is the equation of simple harmonic motion. The similarity of the equations $\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}$ and $\ddot{y} = ky$ is not simply a matter of mathematical formalism. For suitable matrices $\mathbf{A}$, the vector equation $\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}$ does describe a generalization of simple harmonic motion to motion in more than one dimension. The modelling of mechanical systems displaying such motion is discussed in *Unit 24* on normal modes. Our concern here is with the mathematical solution of the equation $\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}$. However, in the programme we take an informal look at an example of two-dimensional motion—the ball in the bowl. This example is provided to help you to visualize the solution of the equation $\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}$ which is obtained in the programme.

Simple harmonic motion was discussed in *Unit 7*.

The ball actually moves in three directions but its motion is completely determined by the behaviour of its two horizontal coordinates x and y.

The solution of the equation involves knowledge of the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. I will ask you to find these before viewing the programme.

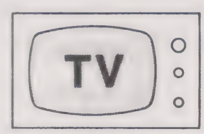
Exercise 1

Calculate the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -\frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix}.$$

[*Solution on p. 35*]

Now view the television programme 'Two-way motion'.



TV 22

Post-programme notes

To solve a system of the form

$$\ddot{\mathbf{r}} = \mathbf{B}\mathbf{r} \tag{2}$$

we may look for solutions of the form

$$\mathbf{r} = \mathbf{a}e^{\mu t} \tag{3}$$

We are not using exactly the same notation as in the programme.

where $\mathbf{a}$ is a constant vector, and $\mathbf{a} \neq \mathbf{0}$. With $\mathbf{r}$ given by Equation (3), we have

$$\dot{\mathbf{r}} = \mu\mathbf{a}e^{\mu t} \quad \text{and} \quad \ddot{\mathbf{r}} = \mu^2\mathbf{a}e^{\mu t}.$$

Hence (3) is a solution of $\ddot{\mathbf{r}} = \mathbf{B}\mathbf{r}$ if, and only if,

$$\mu^2\mathbf{a}e^{\mu t} = \mathbf{B}\mathbf{a}e^{\mu t};$$

i.e. $\quad \mu^2\mathbf{a} = \mathbf{B}\mathbf{a}.$

With $\mathbf{a} \neq \mathbf{0}$, this condition asserts that μ^2 is an eigenvalue of the matrix $\mathbf{B}$, and $\mathbf{a}$ is a corresponding eigenvector.

If λ is an eigenvalue of $\mathbf{B}$ and $\mathbf{a}$ is a corresponding eigenvector, we have two solution of $\ddot{\mathbf{r}} = \mathbf{B}\mathbf{r}$ of the form $\mathbf{r} = \mathbf{a}e^{\mu t}$. These are

$$\mathbf{r} = \mathbf{a}e^{\sqrt{\lambda}\, t} \text{ and } \mathbf{r} = \mathbf{a}e^{-\sqrt{\lambda}\, t}.$$

In the programme, we concentrated on the pair of Equations (1) on page 19, which are equivalent to $\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}$ with $\mathbf{r} = [x \quad y]^T$ and

$$\mathbf{A} = \begin{bmatrix} -\frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix}. \tag{4}$$

The eigenvalues and eigenvectors of **A** are given in the solution of Exercise 1, where we found that:

$[1 \quad -1]^T$ is an eigenvector corresponding to the eigenvalue -1;

$[1 \quad 1]^T$ is an eigenvector corresponding to the eigenvalue $-\frac{1}{9}$.

So, the solutions of $\ddot{\mathbf{r}} = \mathbf{Ar}$ given by the eigenvalue -1 are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{it} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-it}.$$

More generally, a linear combination of these solutions is also a solution. That is

$$\mathbf{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (c_1 e^{it} + d_1 e^{-it}),$$

where c_1 and d_1 are constants, is a solution. Use of Euler's formula enables us to rewrite this as

$$\mathbf{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_1 \cos t + D_1 \sin t) \tag{5}$$

Euler's formula was discussed in *Unit 5*. It states that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

where C_1 and D_1 are arbitrary real constants. Similarly, corresponding to the eigenvalue $-\frac{1}{9}$, we have a solution

$$\mathbf{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (c_2 e^{(1/3)it} + d_2 e^{(-1/3)it})$$

or

$$\mathbf{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_2 \cos \tfrac{1}{3}t + D_2 \sin \tfrac{1}{3}t). \tag{6}$$

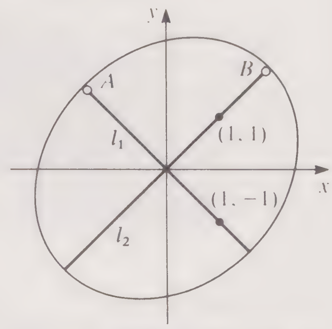


Figure 2. If the ball-bearing is released from either of the positions *A* or *B*, then the Equations (1) imply simple harmonic oscillations along the straight lines l_1 or l_2 respectively. Such straight line motions were observed in the real bowl.

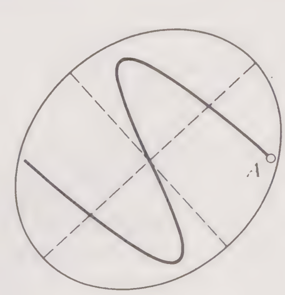
Although (5) and (6) contain arbitrary constants, neither is the *general* solution of $\ddot{\mathbf{r}} = \mathbf{Ar}$. The general solution is a linear combination of both (5) and (6). That is

$$\mathbf{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_1 \cos t + D_1 \sin t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_2 \cos \tfrac{1}{3}t + D_2 \sin \tfrac{1}{3}t). \tag{7}$$

The programme showed computer animations of the motion of a point in a plane, where the position coordinates (x, y) of the point satisfy the original pair of differential equations (1) (or, equivalently, $\ddot{\mathbf{r}} = \mathbf{Ar}$ with **A** given by (4) and $\mathbf{r} = [x \quad y]^T$).

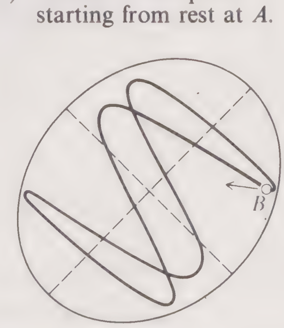
Each of the Solutions (5) and (6) above corresponds to a particularly interesting motion of such a point. Looking at (5) we see that it gives the position $\mathbf{r} = [x \quad y]^T$ as a scalar multiple $(C_1 \cos t + D_1 \sin t)$ of a fixed vector $[1 \quad -1]^T$. So, in the motion described by (5), the point moves along a straight line defined by the vector $[1 \quad -1]^T$ (see Figure 2). The factor $C_1 \cos t + D_1 \sin t$ varies sinusoidally, so that the point oscillates in simple harmonic motion along this line.

Similarly Equation (6) implies simple harmonic motion along the line defined by the vector $[1 \quad 1]^T$. Notice also that Equations (5) and (6) show that the angular frequencies of these two special motions are in the ratio 1:3.



(a) Motion of the point starting from rest at *A*.

The motions described by Equations (5) and (6) are only achieved under suitable initial conditions. The general motion of the point, under arbitrary initial conditions, is given by the general solution (7) of the equation $\ddot{\mathbf{r}} = \mathbf{Ar}$. But any solution of the form (7) is just the sum of a solution of the form (5) and one of the form (6). Thus any motion of the point can be seen as the sum of two special motions, one along each of the lines defined by $[1 \quad 1]^T$ and $[1 \quad -1]^T$. The programme included a computer animation illustrating this fact. Figure 3 shows examples of the paths (orbits) that such a point may follow. The general motion described by the Equations (1), or their solution (7), is of the point tracing and retracing orbits of the form shown in Figure 3.



(b) Motion of the point starting from *B* in the direction shown.

Figure 3. Examples of orbits determined by the Equations (1).

Now consider the real bowl. By viewing the bowl from above we observe the behaviour of the horizontal x, y co-ordinates of the ball bearing. By comparing

this view of the bowl with the computer animations we can judge how accurately Equations (1) describe the motion of the ball bearing. In the programme we found that the ball bearing could be made to describe oscillation along the lines defined by $[1 \quad 1]^T$ and $[1 \quad -1]^T$. However, the oscillations were damped rather than pure simple harmonic motion. Furthermore, the general orbits of the ball bearing in the real bowl were not of the form shown in Figure 3, for as well as damping, there were other inaccuracies in the real bowl. One was that the frequencies of the two special motions were not exactly in the ratio 1:3. We concluded the programme by trying to adjust the mathematical solution so as to make it give a more accurate description of the behaviour of the real ball-bearing. We did this by looking at computer animations obtained by adding simple harmonic motions in the same two directions $[1 \quad 1]^T$ and $[1 \quad -1]^T$ but with frequencies in a ratio which differed slightly from 1:3 (see Figure 4).

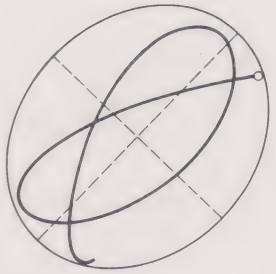


Figure 4. By adjusting the ratio of the animation's frequencies in the special directions, we can obtain a better correspondence between the animation and the motion in the real bowl.

Exercise 2

An object moves so that its position, $\mathbf{r} = [x \quad y]^T$, in a plane satisfies the equations

$$\ddot{x} = -\tfrac{25}{7}x + \tfrac{6}{7}y$$

$$\ddot{y} = \tfrac{9}{7}x - \tfrac{10}{7}y.$$

Give the directions (if any) in which such an object can describe simple harmonic motion in a straight line, and the frequency of any such motion.

[*Solution on p. 35*]

3.2 Solution of second-order systems of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$

In the previous subsection we saw that

$$\mathbf{x} = \mathbf{a}e^{\mu t} \tag{4}$$

is a solution of

$$\ddot{\mathbf{x}} = \mathbf{Bx} \tag{3}$$

so long as μ^2 is an eigenvalue of $\mathbf{B}$ and $\mathbf{a}$ is a corresponding eigenvector. For each non-zero eigenvalue λ of $\mathbf{B}$, we have two solutions of (3): $\mathbf{a}e^{\sqrt{\lambda}t}$ and $\mathbf{a}e^{-\sqrt{\lambda}t}$. Aside from the fact that there are more terms in the solution, solving a second-order system of the form (3) is no harder than solving a first-order system. The general solution of (3) can usually be built up from solutions of the form (4). But as in Subsection 1.2 this method does not invariably work—it requires that the matrix $\mathbf{B}$ has enough eigenvectors. The following theorem and procedure for second-order systems of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$ are very similar to those in Subsection 1.2 for first-order systems.

Theorem 1

Let $\mathbf{B}$ be an $n \times n$ matrix. Suppose that $\mathbf{B}$ has n linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

If all the eigenvalues are non-zero the general solution of the system of differential equations

$$\ddot{\mathbf{x}} = \mathbf{Bx}$$

is

$$\mathbf{x} = \sum_{r=1}^{n} \mathbf{a}_r(c_r e^{\sqrt{\lambda_r}t} + d_r e^{-\sqrt{\lambda_r}t}) \tag{5}$$

where c_r and d_r are arbitrary constants $(1 \le r \le n)$.

If any eigenvalue λ_r is zero the general solution is obtained by replacing the rth term in (5) by $(c_r + d_r t)$.

In the example studied in the previous subsection the eigenvalues were negative and we had to express the general solution in terms of real functions. In general, the real form of the terms $c_r e^{\sqrt{\lambda_r}t} + d_r e^{-\sqrt{\lambda_r}t}$ in (5) depend on the nature of the

eigenvalues. There are three cases to consider: (1) λ_r real and positive, (2) λ_r real and negative and (3) λ_r complex. (The case $\lambda_r = 0$ is considered separately in Theorem 1.)

Case 1: λ_r real and positive

In this case the square roots of λ_r are real and the term in (5),

$$\mathbf{a}_r(c_r e^{\sqrt{\lambda_r}\,t} + d_r e^{-\sqrt{\lambda_r}\,t}),$$

is already explicitly real.

Case 2: λ_r real and negative

Now the square roots of λ_r can be written as

$$i\sqrt{-\lambda_r} \qquad \text{and} \qquad -i\sqrt{-\lambda_r}.$$

Since both $e^{i\sqrt{-\lambda_r}\,t}$ and $e^{-i\sqrt{-\lambda_r}\,t}$ are combinations of $\cos\sqrt{-\lambda_r}\,t$ and $\sin\sqrt{-\lambda_r}\,t$, the term in (5) can in this case be written as

$$\mathbf{a}_r(C_r \cos\sqrt{-\lambda_r}\,t + D_r \sin\sqrt{-\lambda_r}\,t).$$

where C_r and D_r are combinations of c_r and d_r and are again arbitrary constants.

Case 3: λ_r complex

In this case we can replace the complex exponentials in (5) in a way similar to that used in Procedure 1.2(a) for homogeneous first-order systems.

We know from Theorem 1 that if λ is a complex eigenvalue of $\mathbf{B}$ with corresponding eigenvector $\mathbf{a}$ then the functions

$$\mathbf{a}e^{\sqrt{\lambda}\,t} \text{ and } \mathbf{a}e^{-\sqrt{\lambda}\,t} \tag{6}$$

will appear in the general solution (5). It turns out that if the matrix $\mathbf{B}$ is real then $\bar{\lambda}$ will also be an eigenvalue of $\mathbf{B}$, and $\bar{\mathbf{a}}$ will be a corresponding eigenvector. This implies that the functions

$$\bar{\mathbf{a}}e^{\sqrt{\bar{\lambda}}\,t} \text{ and } \bar{\mathbf{a}}e^{-\sqrt{\bar{\lambda}}\,t} \tag{7}$$

will also appear in the general solution. Now the functions (7) are complex conjugates of the functions (6) and so we can replace all four functions by the *real* functions

$$\operatorname{Re}(\mathbf{a}e^{\sqrt{\lambda}\,t}), \qquad \operatorname{Im}(\mathbf{a}e^{\sqrt{\lambda}\,t}), \qquad \operatorname{Re}(\mathbf{a}e^{-\sqrt{\lambda}\,t}), \qquad \text{and} \qquad \operatorname{Im}(\mathbf{a}e^{-\sqrt{\lambda}\,t}).$$

To summarize, we have the following procedure:

Procedure 3.2:
To solve a linear, second-order, system of the form $\ddot{\mathbf{x}} = \mathbf{B}\mathbf{x}$

For this procedure to work $\mathbf{B}$ must be an $n \times n$ matrix with n linearly independent eigenvectors.

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{B}$ and a corresponding set of linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.
2. Write down the general solution in real form. There are four cases to consider:

   (i) If all the eigenvalues are real and positive the general solution is

   $$\mathbf{x} = \sum_{r=1}^{n} \mathbf{a}_r(C_r e^{\sqrt{\lambda_r}\,t} + D_r e^{-\sqrt{\lambda_r}\,t}). \tag{8}$$

   (ii) If $\lambda_r = 0$ replace the rth term in (8) by

   $$\mathbf{a}_r(C_r + D_r t).$$

   (iii) If λ_r is real and negative replace the rth term in (8) by

   $$\mathbf{a}_r(C_r \cos\sqrt{-\lambda_r}\,t + D_r \sin\sqrt{-\lambda_r}\,t).$$

   (iv) If an eigenvalue λ is complex replace the functions $\mathbf{a}e^{\sqrt{\lambda}\,t}$, $\mathbf{a}e^{-\sqrt{\lambda}\,t}$, $\bar{\mathbf{a}}e^{\sqrt{\bar{\lambda}}\,t}$ and $\bar{\mathbf{a}}e^{-\sqrt{\bar{\lambda}}\,t}$ in (8) by the functions

   $$\operatorname{Re}(\mathbf{a}e^{\sqrt{\lambda}\,t}), \operatorname{Im}(\mathbf{a}e^{\sqrt{\lambda}\,t}), \operatorname{Re}(\mathbf{a}e^{-\sqrt{\lambda}\,t}) \text{ and } \operatorname{Im}(\mathbf{a}e^{-\sqrt{\lambda}\,t}).$$

The eigenvalues of the matrix **B** do not necessarily all fall into the same case, of course. The example below illustrates this.

Example 1

Find the general solution of the system

$$\begin{aligned} \ddot{x}_1 &= x_1 + 3x_2 - x_3 \\ \ddot{x}_2 &= 3x_1 + x_2 + x_3 \\ \ddot{x}_3 &= 4x_3. \end{aligned}$$

Solution

We have a system of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$ with

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

The first step is to find the eigenvalues and eigenvectors of **B**.

The eigenvalues can be found by solving the characteristic equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. We have

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 3 & -1 \\ 3 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$\begin{aligned} &= (4-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)((1-\lambda)^2 - 3^2) \\ &= -(4-\lambda)(4-\lambda)(2+\lambda). \end{aligned}$$

So the eigenvalues are -2 and 4 (repeated).

The eigenvectors corresponding to the eigenvalue -2 satisfy

$$\begin{bmatrix} 3 & 3 & -1 \\ 3 & 3 & 1 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $w = 0$ and $u = -v$, and so an eigenvector is $[1 \quad -1 \quad 0]^T$. The eigenvectors corresponding to the eigenvalue 4 satisfy

$$\begin{bmatrix} -3 & 3 & -1 \\ 3 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These equations reduce to

$$-3u + 3v - w = 0.$$

Since we effectively have only one equation in three unknowns, we can find two linearly independent eigenvectors corresponding to the eigenvalue 4: for example $[1 \quad 1 \quad 0]^T$ and $[0 \quad 1 \quad 3]^T$.

By Procedure 3.2 the general solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}(C_1e^{2t} + D_1e^{-2t}) + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}(C_2e^{2t} + D_2e^{-2t}) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}(C_3\cos\sqrt{2}\,t + D_3\sin\sqrt{2}\,t)$$

or alternatively

$$x_1 = C_1 e^{2t} + D_1 e^{-2t} + C_3 \cos\sqrt{2}\,t + D_3 \sin\sqrt{2}\,t$$
$$x_2 = (C_1 + C_2)e^{2t} + (D_1 + D_2)e^{-2t} - C_3 \cos\sqrt{2}\,t - D_3 \sin\sqrt{2}\,t$$
$$x_3 = 3(C_2 e^{2t} + D_2 e^{-2t}).$$

Exercise 3

Find the general solution of each of the systems below.

(i) $\ddot{x}_1 = x_2$
$\ddot{x}_2 = x_1$

(ii) $\ddot{x}_1 = -10x_1 - 6x_2$
$2\ddot{x}_2 = -12x_1 - 20x_2.$

Note the 2 on the left-hand side of part (ii).

[*Solution on p. 35*]

Summary of Section 3

To solve a second-order, constant-coefficient, system of the form $\ddot{\mathbf{x}} = \mathbf{Bx}$, for suitable $\mathbf{B}$ we may proceed in a similar way to that described in Section 1 for a first-order system. The details are given in Procedure 3.2.

Of particular significance in mechanics is the case where the eigenvectors of $\mathbf{B}$ are real and negative. In this case the model $\ddot{\mathbf{x}} = \mathbf{Bx}$ represents a generalization of simple harmonic motion (*Unit 7*). In the television programme we looked at an example of this: the two-dimensional motion of an object whose position vector $\mathbf{r} = [x \quad y]^T$ varied with time according to the system of differential equations $\ddot{\mathbf{r}} = \mathbf{Ar}$ where the matrix $\mathbf{A}$ had negative eigenvalues.

4 Forced oscillations

4.1 The structure of the solution

In the unit 'Differential equations II' we introduced certain results about the solution of the forced oscillation equation

Unit 6, Section 4

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = b_1 \cos vt + b_2 \sin vt.$$

These results were used extensively in the mechanics unit on 'Damped and forced vibrations'.

Unit 8

In this section, we discuss the extension of these results to *systems* of differential equations. I will first remind you of the results for a single equation, by looking at an example.

Example 1

We consider the solution of the equation

$$\ddot{x} + 2\dot{x} + 5x = 2\cos 3t. \qquad (1)$$

To solve this equation we first calculate the complementary function; that is, the general solution of the associated homogeneous equation

$$\ddot{x} + 2\dot{x} + 5x = 0.$$

I will not give details of this calculation. The complementary function is

$$x_c = e^{-t}(A\cos 2t + B\sin 2t).$$

Next we calculate a particular solution. We can do this by the method of phasors, in which we try $x = \mathrm{Re}(ze^{3it})$ and calculate the complex number z. Since the phasor of $2\cos 3t$ is 2, we get here

$$((3i)^2 + 2(3i) + 5)z = 2,$$

and so

$$z = \frac{2}{6i - 4} = -\frac{1}{13}(2 + 3i).$$

Thus a particular solution is

$$x_p = -\tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin 3t.$$

The general solution of (1) is found by adding the particular solution and the complementary function, and is

$$\begin{aligned} x &= x_c + x_p \\ &= e^{-t}(A\cos 2t + B\sin 2t) - \tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin 3t. \end{aligned}$$

Now let us consider the behaviour of this solution for large values of t. The complementary function tends to zero for large t, because of the term e^{-t}. Thus for large t *any* solution of (1) is approximately equal to the particular solution

$$x_p = -\tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin 3t.$$

This is known as the **steady-state** solution of (1). We refer to the complementary function as a **transient**, since its effects die out as t becomes large. We saw in *Unit 8* that in many mechanical problems on forced vibrations, a knowledge of the steady-state solution is all that is needed, so long as we know that the complementary function is indeed a transient (i.e. it tends to zero for large t).

In general, we have the following important information about the solution of the single equation

$$a_1\ddot{x} + a_2\dot{x} + a_3x = b_1\sin\omega t + b_2\cos\omega t. \qquad (2)$$

(i) The general solution of (2) is the sum of the complementary function and any one particular solution.

(ii) So long as each of a_1, a_2 and a_3 is positive, the complementary function is a transient (i.e. it tends to zero for large t).

(iii) There exists one purely sinusoidal particular solution, to which any solution converges for large t.

Now the motion of a system of vibrating particles under an external force may be modelled by a system of differential equations, of the form

We shall study such systems in *Unit 24*.

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{b}_1\cos\omega t + \mathbf{b}_2\sin\omega t. \qquad (3)$$

So analysis for this system parallel to that above for the single equation (2) would be very informative. To achieve this we need three things.

(i) To know that the general solution of (3) is made up of 'particular solution plus complementary function'.

(ii) To have a condition on the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ which ensures that the complementary function is a transient.

(iii) To know that there is a purely sinusoidal solution of (3), and how to calculate it.

Each of these can be dealt with, and we will do so now. Points (i) and (ii) are resolved by Theorems 1 and 2 below. These are stated without proof, although a proof of Theorem 1 would be quite straightforward. The proof of Theorem 2 depends on methods in linear algebra beyond the scope of this course.

Theorem 1

Suppose that $\mathbf{x}_p$ is a particular solution of the system

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{h}(t). \qquad (4)$$

Then the general solution of (4) is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_c$$

where $\mathbf{x}_c$ (the **complementary function**) is the general solution of the associated homogeneous system

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{0}.$$

Theorem 2

Suppose that each of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ has constant real entries, is symmetric, and has the property that each of its eigenvalues is positive. Then any solution of the system

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{0}$$

has the property that $\mathbf{x}$ tends to $\mathbf{0}$ as t becomes large. (That is, every component of the vector $\mathbf{x}$ becomes small as t becomes large.)

A symmetric matrix $\mathbf{A}$ is one for which $\mathbf{A}^T = \mathbf{A}$ (see *Unit 20*, Subsection 4.4).

Exercise 1

Which of the systems below satisfies the criteria given in Theorem 2 for the solution to be a transient?

(i) $\ddot{x}_1 + 3\dot{x}_1 + 2\dot{x}_2 + 2x_1 - x_2 = 0$
$3\ddot{x}_2 + 2\dot{x}_1 + 3\dot{x}_2 - x_1 + 2x_2 = 0$

(ii) $\ddot{x}_1 + 3\dot{x}_1 + 2\dot{x}_2 + 2x_1 = 0$
$2\ddot{x}_2 + 2\dot{x}_1 + 3\dot{x}_2 = 0$

(iii) $\ddot{x}_1 + \dot{x}_1 = 2x_1 + x_2$
$\ddot{x}_2 + \dot{x}_2 = x_1 + 2x_2.$

[*Solution on p. 36*]

Exercise 2

A particular solution of

$$\ddot{x}_1 = x_2 + t$$

$$\ddot{x}_2 = x_1 + 1$$

is not difficult to spot. Do so, and hence find the general solution of the system. (You may reuse the solution of Exercise 3(i) of Section 3 here.)

[*Soluton on p. 36*]

4.2 Finding a sinusoidal particular solution

In (iii) above, we wished to know a purely sinusoidal solution of a system of equations of the form (3). In this subsection we shall see how such a solution can be calculated. The method uses phasors and is an extension of the procedure in *Unit 6* for finding a sinusoidal particular solution of a single second-order differential equation. We look for a solution of the form $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{i\omega t})$ where $\mathbf{z}$ is a constant vector with complex entries. Let us see an example.

Example 2

Find a sinusoidal particular solution of the system

$$\ddot{x}_1 + \dot{x}_2 = \sin 2t$$

$$\ddot{x}_2 + \dot{x}_1 + 2x_2 = \cos 2t.$$

Solution

Step 1: First we write the right-hand sides in phasor form:

Phasors were defined in *Unit 5* and used in *Units 6* and *8*.

$$\begin{aligned} \ddot{x}_1 + \dot{x}_2 &= \mathrm{Re}(-ie^{2it}) \\ \ddot{x}_2 + \dot{x}_1 + 2x_2 &= \mathrm{Re}(e^{2it}). \end{aligned} \tag{5}$$

Step 2: Next we look for a solution of the form

$$\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{2it}).$$

Then if $\mathbf{x} = [x_1 \quad x_2]^T$ and $\mathbf{z} = [z_1 \quad z_2]^T$, we have

$$x_1 = \mathrm{Re}(z_1 e^{2it}),$$

so $\quad \dot{x}_1 = \mathrm{Re}(2iz_1 e^{2it}),$

and $\quad \ddot{x}_1 = \mathrm{Re}(-4z_1 e^{2it});$

and similarly for x_2.

Thus to satisfy (5) we need

$$-4z_1 + 2iz_2 = -i \tag{6}$$

$$-4z_2 + 2iz_1 + 2z_2 = 1. \tag{7}$$

Equation (7) may be rewritten as

$$2iz_1 - 2z_2 = 1. \tag{8}$$

Step 3: Thus we have a pair of simultaneous equations, which may be solved by Gaussian elimination, as follows. Adding $2i/4$ times (6) to (8) gives

$$-3z_2 = \tfrac{3}{2}$$

i.e. $\quad z_2 = -\tfrac{1}{2}.$

Substituting in (6) gives

$$-4z_1 - i = -i$$

i.e. $\quad z_1 = 0.$

Step 4: Now z_1 and z_2 are the phasors of the required particular solution. The solution is therefore

$$x_1 = 0$$

$$x_2 = -\tfrac{1}{2}\cos 2t.$$

The method used in the example is set out in the following procedure.

Procedure 4.2

To find a sinusoidal particular solution of

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{b}_1\cos\omega t + \mathbf{b}_2\sin\omega t.$$

1. Write the system in the form

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathrm{Re}(\mathbf{d}e^{i\omega t}).$$

2. Look for a solution of the form

$$\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{i\omega t}),$$

where $\mathbf{z} = [z_1 \quad z_2 \quad \ldots \quad z_n]^T$ and each z_r is a complex number. This leads to a system of simultaneous linear equations for the z_r's.

3. Solve this system of linear equations, and so find $\mathbf{z}$.

4. The required solution is then

$$\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{i\omega t}).$$

The only significant difference between this procedure and that in *Unit 6* for a single equation is that here, in Step 3, we must solve a set of simultaneous linear equations, rather than just a single equation.

Let us see what happens if we apply Procedure 4.2 in general. Suppose we have written our system in the form

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathrm{Re}(\mathbf{d}e^{i\omega t})$$

where $\mathbf{d}$ is some constant, complex, vector. We look for a solution of the form $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{i\omega t})$. Then $\dot{\mathbf{x}} = \mathrm{Re}(i\omega\mathbf{z}e^{i\omega t})$ and $\ddot{\mathbf{x}} = \mathrm{Re}(-\omega^2\mathbf{z}e^{i\omega t})$. So we have a solution so long as

$$\mathbf{A}_1(-\omega^2\mathbf{z}e^{i\omega t}) + \mathbf{A}_2(i\omega\mathbf{z}e^{i\omega t}) + \mathbf{A}_3(\mathbf{z}e^{i\omega t}) = \mathbf{d}e^{i\omega t},$$

i.e. $\quad (-\omega^2\mathbf{A}_1 + i\omega\mathbf{A}_2 + \mathbf{A}_3)\mathbf{z} = \mathbf{d}.$

This represents a set of n simultaneous equations (with complex coefficients) in n unknowns (compare with Equations (6) and (7) in Example 2). These equations will usually have a unique solution for $\mathbf{z}$, and so we will obtain a sinusoidal particular solution $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{i\omega t})$.

There could be problems with this method if the matrix $-\omega^2\mathbf{A}_1 + i\omega\mathbf{A}_2 + \mathbf{A}_3$ is singular. (This compares with the problem of finding a particular solution of, for example, the single equation

$$\ddot{x} + 4x = \sin 2t,$$

which has no solution of the form $x = \text{Re}(ze^{2it})$.) So, once again, we have a procedure that will usually work, but which could break down. I will not get into a discussion of how to deal with possible 'problem cases' here.

The question below involves the use of Procedure 4.2. You may wish to treat it as an Exercise, or as another worked Example, before going on to try Exercise 3 for yourself.

Question

Find a sinusoidal solution of

$$\begin{aligned} \ddot{x}_1 + 8\dot{x}_2 + x_1 &= 8\sin 3t - 8\cos 3t \\ \ddot{x}_2 + \dot{x}_1 &= -6\sin 3t - 3\cos 3t. \end{aligned}$$

Solution

Step 1: First we write the right-hand side in the appropriate form

$$\begin{aligned} \ddot{x}_1 + 8\dot{x}_2 + x_1 &= \text{Re}(-8ie^{3it} - 8e^{3it}) = \text{Re}\left((-8i - 8)e^{3it}\right) \\ \ddot{x}_2 + \dot{x}_1 &= \text{Re}\left((6i - 3)e^{3it}\right). \end{aligned}$$

Step 2: Next we look for a solution of the form $\mathbf{x} = \text{Re}(\mathbf{z}e^{3it})$, where $\mathbf{x} = [x_1 \quad x_2]^T$ and $\mathbf{z} = [z_1 \quad z_2]^T$. This is a solution if

$$\begin{aligned} (3i)^2 z_1 + 8(3i)z_2 + z_1 &= -8i - 8 \\ (3i)^2 z_2 + 3iz_1 &= 6i - 3. \end{aligned}$$

That is, if

$$\begin{aligned} -8z_1 + 24iz_2 &= -8i - 8 \\ 3iz_1 - 9z_2 &= 6i - 3. \end{aligned}$$

Step 3: Multiplying the first equation by $3i/8$, and adding this to the second equation, gives

$$-18z_2 = 3i.$$

i.e. $\qquad z_2 = -\frac{1}{6}i.$

Substituting this into the first equation gives

$$-8z_1 + 4 = -8i - 8.$$

i.e. $\qquad z_1 = \frac{3}{2} + i.$

Step 4: Now z_1 and z_2 are the phasors of the required particular solution. The particular solution is therefore

$$\begin{aligned} x_1 &= \tfrac{3}{2}\cos 3t - \sin 3t \\ x_2 &= \tfrac{1}{6}\sin 3t. \end{aligned}$$

(As usual, this solution should be checked by substitution in the original equations.)

Exercise 3

Find a sinusoidal particular solution of each of the systems below.

(i) $\ddot{x}_1 + 4x_1 + 2x_2 = 6\cos 2t$
$\quad\ \ddot{x}_2 + x_1 + 9x_2 = 2\sin 2t$

(ii) $\ddot{x}_1 + 2\ddot{x}_2 + \dot{x}_1 + x_1 - 3x_2 = \sin t$
$\quad\ 3\ddot{x}_1 + \ddot{x}_2 + 2\dot{x}_2 + 2x_1 + x_2 = \cos t - 2\sin t.$

[*Solution on p. 36*]

Summary of Section 4

We discussed results about the solution of the system

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{b}_1 \cos \omega t + \mathbf{b}_2 \sin \omega t \tag{3}$$

relevant to the motion of a system of vibrating particles. The results are similar to those for a single second-order equation. The general solution of (3) is the sum of any one particular solution and the **complementary function**: that is, the general solution of $\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{0}$. We saw that if each of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ has constant real entries, is symmetric, and has all its eigenvectors positive, then the complementary function is a **transient**; that is, all the entries in the vector $\mathbf{x}$ tend to zero for large t. We saw also that (3) has a particular solution of the form $\mathbf{x} = \mathbf{d}_1 \cos \omega t + \mathbf{d}_2 \sin \omega t$. This can be calculated by looking for a solution of (3) of the form $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{i\omega t})$ and calculating the complex vector $\mathbf{z}$ from the resulting linear equations, as described in Procedure 4.2.

Thus, so long as the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are of the appropriate form, any solution of (3) settles down to the same **steady-state** solution in the long-term. This is independent of the initial conditions, which only affect the transient term.

5 End of unit exercises and problems

Exercises 1–3 below cover the main objectives of the unit. They can be used either for further practice, or when revising as a check that you have covered the central ideas of the unit.

Problems 1 and 2 take the ideas in the unit a little further.

Exercise 1

Find the general solution of each of the systems below.

(i) $\dot{x}_1 = 8x_1 - 5x_2$
$x_2 = 10x_1 - 7x_2$

(ii) $\dot{x}_1 = 8x_1 - 5x_2 + 2e^t$
$\dot{x}_2 = 10x_1 - 7x_2 - e^{2t}$

(iii) $\dot{x}_1 = 6x_1 - 4x_2$
$\dot{x}_2 = 10x_1 - 6x_2$

(iv) $\ddot{x}_1 = 8x_1 - 5x_2$
$\ddot{x}_2 = 10x_1 - 7x_2$

(v) $\ddot{x}_1 = 8x_1 - 5x_2 + \sin 2t$
$\ddot{x}_2 = 10x_1 - 7x_2 + 2\cos 2t$

(vi) $\dot{x}_1 = -x_1 + 2x_2$
$\dot{x}_2 = -x_1 - 3x_2$
$\dot{x}_3 = x_2 - 4x_3$

(vii) $\dot{x}_1 = 4x_1 + 4x_2 - \dot{x}_2$
$3x_1 + x_2 = -x_3 + \dot{x}_2$
$\dot{x}_3 - 4x_3 = 0.$

[*Solution on p. 36*]

Exercise 2

Describe the behaviour of the solutions of the following system for large values of t.

$$4\ddot{x}_1 + \ddot{x}_2 + 3\dot{x}_1 + 3x_1 + 4x_2 = \cos 2t$$
$$\ddot{x}_1 + 4\ddot{x}_2 + 3\dot{x}_2 + 4x_1 + 6x_2 = \sin 2t.$$

[*Solution on p. 39*]

Exercise 3

Construct a set of recurrence relations, based on Euler's method, suitable for generating approximations to the solution of the following system.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_1 + 3x_2 - tx_3 - \cos t,$$

where $x_1(0) = 2, \quad x_2(0) = -1, \quad x_3(0) = 4.$

[*Solution on p. 39*]

Problem 1

(i) Suppose that $y = x_1$ and that x_1 satisfies the system given in Exercise 3. Show that y satisfies the differential equation

$$\dddot{y} + t\ddot{y} - 3\dot{y} - y = -\cos t.$$

(ii) Can the process in (i) be reversed? Given that y satisfies the differential equation

$$\dddot{y} + 4ty = \sin 2t,$$

with $y(0) = 1$, $\dot{y}(0) = 2$, $\ddot{y}(0) = 0$, can you construct a first-order system that y must satisfy?

Use this idea to construct a set of recurrence relations to generate numerical solutions to this differential equation.

(iii) The idea introduced in (ii) above can be extended to systems of higher-order equations. Consider, for example, the pair of second-order (non-linear) equations below, which model the motion of a particle in two dimensions under the influence of gravity, and of a resisting force whose size is proportional to the square of the velocity.

$$\ddot{x} = \frac{-kx}{x^2 + y^2} - \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}\dot{x}$$

$$\ddot{y} = \frac{-ky}{x^2 + y^2} - \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}\dot{y}.$$

Here k and λ are constants.

By introducing new variables $x_1 = x$, $x_2 = y$, $x_3 = \dot{x}$, $x_4 = \dot{y}$, construct a first-order system equivalent to the second-order system.

Hence construct a set of recurrence relations that would generate numerical solutions to this second-order system (for given values of k and λ).

[*Solution on p. 40*]

Problem 2 (*This introduces methods that cover the special cases not dealt with by the Procedures in Sections 1 and 2.*)

(i) Show that $\mathbf{x} = \mathbf{a}e^{\lambda t} + \mathbf{b}te^{\lambda t}$ is a solution of the system $\dot{\mathbf{x}} = \mathbf{Bx}$ so long as

(a) $\mathbf{b}$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue λ, and

(b) $\mathbf{a}$ satisfies $(\mathbf{B} - \lambda\mathbf{I})\mathbf{a} = \mathbf{b}$.

(ii) (a) Try to solve the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 + 4x_2$$

by Procedure 1.2 of Section 1. What goes wrong?

(b) Use part (i) to find two linearly independent solutions of this system.

(c) Write down the general solution of the system. (You may assume that it is a linear combination of the two solutions you have found.)

(iii) Can you find a solution of the form specified in (i) for the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 + 3x_2?$$

(iv) Solve the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 + 4x_2 + e^t,$$

using the steps outlined below.

(a) In part (ii) you calculated an eigenvector $\mathbf{b}$ of the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$$

and a vector $\mathbf{a}$ such that $(\mathbf{B} - \lambda\mathbf{I})\mathbf{a} = \mathbf{b}$. Form a matrix $\mathbf{P}$ whose columns are $\mathbf{b}$ and $\mathbf{a}$ (in that order). Show that

$$\mathbf{P}^{-1}\mathbf{BP} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

(b) Make the substitution $\mathbf{x} = \mathbf{Py}$ in the given system, to obtain

$$\dot{y}_1 = 2y_1 + y_2$$
$$\dot{y}_2 = 2y_2 + e^t.$$

Solve the second of these equations for y_2, and hence solve the first equation.

(c) Hence solve the given system.

[*Solution on p. 40*]

Appendix 1: Solutions to the exercises

Solutions to the exercises in Section 1

1. (i) Differentiating the components of $\mathbf{g}(t) = [2t^2 \quad 4 \quad e^{2t}]^T$ we obtain

$$\frac{d\mathbf{g}(t)}{dt} = [4t \quad 0 \quad 2e^{2t}]^T.$$

(ii) Let $\mathbf{a} = [a_1 \quad a_2 \quad \ldots \quad a_n]^T$. Then

$$\mathbf{x} = \mathbf{a}e^{\lambda t} = [a_1e^{\lambda t} \quad a_2e^{\lambda t} \quad \ldots \quad a_ne^{\lambda t}]^T.$$

Thus

$$\begin{aligned}\dot{\mathbf{x}} &= [\lambda a_1e^{\lambda t} \quad \lambda a_2e^{\lambda t} \quad \ldots \quad \lambda a_ne^{\lambda t}]^T\\ &= \lambda e^{\lambda t}[a_1 \quad a_2 \quad \ldots \quad a_n]^T\\ &= \lambda\mathbf{a}e^{\lambda t}.\end{aligned}$$

(This is just the result we would have hoped for.)

2. Systems (1), (2) and (3) are linear, (4) is not. None of (1), (2) and (3) are homogeneous. (1) and (2) are constant-coefficient, (3) is not. (1) and (3) are first-order; (2) is second-order.

3. (i) It is helpful to rewrite the system as

$$\begin{aligned}\dot{x}_1 - 3\dot{x}_2 \qquad &= -2x_1 - x_2\\ \dot{x}_3 &= \quad x_1 + x_2 + x_3\\ -\dot{x}_1 + \dot{x}_2 - \dot{x}_3 &= \quad 0.\end{aligned}$$

In matrix form this is

$$\mathbf{A}\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} \qquad (\text{or } \mathbf{A}\dot{\mathbf{x}} - \mathbf{B}\mathbf{x} = \mathbf{0})$$

with $\mathbf{A} = \begin{bmatrix}1 & -3 & 0\\ 0 & 0 & 1\\ -1 & 1 & -1\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}-2 & -1 & 0\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$.

(ii) It is constant-coefficient and homogeneous. It is first-order.

4. (i) This system is $\mathbf{A}_1\dot{\mathbf{x}} + \mathbf{A}_2\mathbf{x} = \mathbf{h}(t)$ where

$$\mathbf{A}_1 = \begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix}5 & 0\\ 0 & 10\end{bmatrix}, \quad \mathbf{h}(t) = \begin{bmatrix}t\\ 2\end{bmatrix}.$$

We need $\mathbf{A}_1^{-1}$. Since $\mathbf{A}_1$ is 2×2, we can just write this down (see *Unit 20*, Subsection 4.1):

$$\mathbf{A}_1^{-1} = -\tfrac{1}{5}\begin{bmatrix}-1 & -2\\ -2 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & -\frac{1}{5}\end{bmatrix}.$$

In normal form the system is

$$\dot{\mathbf{x}} = -\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{x} + \mathbf{A}_1^{-1}\mathbf{h}(t),$$

where

$$\mathbf{A}_1^{-1}\mathbf{A}_2 = \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & -\frac{1}{5}\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 10\end{bmatrix} = \begin{bmatrix}1 & 4\\ 2 & -2\end{bmatrix},$$

and

$$\mathbf{A}_1^{-1}\mathbf{h}(t) = \begin{bmatrix}\frac{1}{5}t + \frac{4}{5}\\ \frac{2}{5}t - \frac{2}{5}\end{bmatrix}.$$

So the system, in normal form is

$$\dot{\mathbf{x}} = \begin{bmatrix}-1 & -4\\ -2 & 2\end{bmatrix}\mathbf{x} + \begin{bmatrix}\frac{1}{5}t + \frac{4}{5}\\ \frac{2}{5}t - \frac{2}{5}\end{bmatrix}.$$

(ii) This system is $\mathbf{A}_1\dot{\mathbf{x}} + \mathbf{A}_2\mathbf{x} = \mathbf{h}(t)$ where

$$\mathbf{A}_1 = \begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix}5 & 0\\ 1 & -1\end{bmatrix}, \quad \mathbf{h}(t) = \begin{bmatrix}t\\ 0\end{bmatrix}.$$

The rows of $\mathbf{A}_1$ are linearly dependent, so $\mathbf{A}_1$ is singular and $\mathbf{A}_1^{-1}$ does not exist. Hence this system cannot be written in normal form.

(iii) This system is not linear, and hence cannot even be written in matrix terms, let alone normal form.

(iv) This is $\mathbf{A}_1\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}$ where

$$\mathbf{A}_1 = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad \text{and} \quad \mathbf{A}_0 = \begin{bmatrix}t & -1\\ 1 & 2\end{bmatrix}.$$

Now

$$\mathbf{A}_1^{-1} = -\tfrac{1}{2}\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}.$$

so

$$\mathbf{A}_1^{-1}\mathbf{A}_0 = \begin{bmatrix}\frac{1}{2}t + \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2}t - \frac{1}{2} & -\frac{3}{2}\end{bmatrix}.$$

In normal form the system is therefore

$$\dot{\mathbf{x}} = \mathbf{A}_1^{-1}\mathbf{A}_0\mathbf{x}$$

where $\mathbf{A}_1^{-1}\mathbf{A}_0$ is as above.

5. The system is already in the normal form $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ where

$$\mathbf{B} = \begin{bmatrix}2 & 3\\ 2 & 1\end{bmatrix}.$$

We need to find the eigenvalues and eigenvectors of $\mathbf{B}$.

The eigenvalues are found by solving the characteristic equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. Now

$$\begin{aligned}\det(\mathbf{B} - \lambda\mathbf{I}) &= \begin{vmatrix}2-\lambda & 3\\ 2 & 1-\lambda\end{vmatrix}\\ &= (2-\lambda)(1-\lambda) - 6\\ &= \lambda^2 - 3\lambda - 4\\ &= (\lambda - 4)(\lambda + 1).\end{aligned}$$

So the eigenvalues are 4 and -1.

To find the eigenvectors we solve

$$\begin{bmatrix}2-\lambda & 3\\ 2 & 1-\lambda\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = 4$ gives

$$\begin{aligned}-2u + 3v &= 0\\ 2u - 3v &= 0.\end{aligned}$$

So an eigenvector corresponding to the eigenvalue 4 is $[3 \quad 2]^T$.

Case $\lambda = -1$ gives

$$\begin{aligned}3u + 3v &= 0\\ 2u + 2v &= 0.\end{aligned}$$

So an eigenvector corresponding to the eigenvalue -1 is $[1 \quad -1]^T$.

We have two linearly independent eigenvectors, as required. Hence the general solution is

$$\mathbf{x} = C_1\begin{bmatrix}3\\ 2\end{bmatrix}e^{4t} + C_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-t},$$

i.e.

$$\begin{aligned}x_1 &= 3C_1e^{4t} + C_2e^{-t}\\ x_2 &= 2C_1e^{4t} - C_2e^{-t}\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

6. (i) We have $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ with

$$\mathbf{B} = \begin{bmatrix}5 & 4\\ -1 & 0\end{bmatrix}.$$

Now

$$\det(\mathbf{B}-\lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 4 \\ -1 & -\lambda \end{vmatrix}$$
$$= \lambda^2 - 5\lambda + 4$$
$$= (\lambda - 1)(\lambda - 4).$$

So the eigenvalues of $\mathbf{B}$ are 1 and 4.

To find the corresponding eigenvectors we solve

$$\begin{bmatrix} 5-\lambda & 4 \\ -1 & -\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = 1$ gives $u + v = 0$ so an eigenvector corresponding to the eigenvalue 1 is $[1 \quad -1]^T$.

Case $\lambda = 4$ gives $u + 4v = 0$ so an eigenvector corresponding to the eigenvalue 4 is $[-4 \quad 1]^T$.

There are sufficient eigenvectors, so the general solution is

$$\mathbf{x} = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t + C_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}e^{4t},$$

or

$$x_1 = C_1e^t - 4C_2e^{4t}$$
$$x_2 = -C_1e^t + C_2e^{4t}.$$

(ii) We have $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ with

$$\mathbf{B} = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

Now

$$\det(\mathbf{B}-\lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix}$$

$$= (5-\lambda)[(4-\lambda)(-4-\lambda) + 12] + 6[4 + \lambda - 6] - 6[6 - 12 + 3\lambda]$$
$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$
$$= -(\lambda - 1)(\lambda - 2)^2.$$

So the eigenvalues are 1 and 2 (repeated).

To find the corresponding eigenvectors we solve

$$\begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = 1$ gives

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Performing row operations we obtain

$$\begin{bmatrix} 0 & 6 & 2 \\ -1 & 3 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$-u + w = 0$$
$$3v + w = 0.$$

So an eigenvector corresponding to the eigenvalue 1 is $[-3 \quad 1 \quad -3]^T$.

Case $\lambda = 2$ gives

$$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This reduces to the single equation

$$-u + 2v + 2w = 0.$$

Hence there are two linearly independent eigenvectors, for example $[2 \quad 0 \quad 1]^T$ and $[2 \quad 1 \quad 0]^T$.

We have sufficient linearly independent eigenvectors, so the general solution is

$$\mathbf{x} = C_1\begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}e^t + C_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}e^{2t} + C_3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}e^{2t}$$

or

$$x_1 = -3C_1e^t + 2C_2e^{2t} + 2C_3e^{2t}$$
$$x_2 = C_1e^t + C_3e^{2t}$$
$$x_3 = -3C_1e^t + C_2e^{2t}.$$

(iii) We have $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ where

$$\mathbf{B} = \begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix}.$$

Now

$$\det(\mathbf{B}-\lambda\mathbf{I}) = \begin{vmatrix} -2-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$
$$= \lambda^2 + 2\lambda + 2.$$

So there is a conjugate pair of eigenvalues $-1 \pm i$.

The equation for the eigenvector corresponding to the eigenvalue $-1 + i$ is

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$(-1 - i)u + 2v = 0$$
$$-u + (1 - i)v = 0.$$

The first equation is $(1 + i)$ times the second and so both give $2v = (1 + i)u$. Thus an eigenvector corresponding to the eigenvalue $\lambda = -1 + i$ is $\mathbf{a} = [2 \quad 1 + i]^T$. (The eigenvector corresponding to the eigenvalue $\bar{\lambda} = -1 - i$ is therefore $\bar{\mathbf{a}} = [2 \quad 1 - i]^T$, but this need not concern us here.)

Now

$$\mathbf{a}e^{\lambda t} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}e^{(-1+i)t}$$
$$= e^{-t}\begin{bmatrix} 2(\cos t + i\sin t) \\ (1+i)(\cos t + i\sin t) \end{bmatrix}$$
$$= e^{-t}\left(\begin{bmatrix} 2\cos t \\ \cos t - \sin t \end{bmatrix} + i\begin{bmatrix} 2\sin t \\ \cos t + \sin t \end{bmatrix}\right),$$

so

$$\mathrm{Re}(\mathbf{a}e^{\lambda t}) = e^{-t}\begin{bmatrix} 2\cos t \\ \cos t - \sin t \end{bmatrix},$$

$$\mathrm{Im}(\mathbf{a}e^{\lambda t}) = e^{-t}\begin{bmatrix} 2\sin t \\ \cos t + \sin t \end{bmatrix}.$$

Thus, by Procedure 1.2(a), we can write down the general solution in the form

$$\mathbf{x} = e^{-t}\left(C_1\begin{bmatrix} 2\cos t \\ \cos t - \sin t\end{bmatrix} + C_2\begin{bmatrix} 2\sin t \\ \cos t + \sin t\end{bmatrix}\right),$$

that is

$$x_1 = e^{-t}(2C_1\cos t + 2C_2\sin t)$$
$$x_2 = e^{-t}\big((C_1 + C_2)\cos t + (C_2 - C_1)\sin t\big).$$

Solutions to the exercises in Section 2

1. Note that a good deal of the calculation has already been done in Example 1. We have $\dot{\mathbf{x}} = \mathbf{Bx} + \mathbf{h}(t)$ where $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 2 & 1\end{bmatrix}$ as in the Example, but now $\mathbf{h}(t) = [e^{2t} \quad 4e^{2t}]^T$.

Hence Step 1 of Procedure 2.1 is already done. We have

$$\mathbf{P} = \begin{bmatrix} 3 & 1 \\ 2 & -1\end{bmatrix}; \qquad \mathbf{P}^{-1}\mathbf{BP} = \begin{bmatrix} 4 & 0 \\ 0 & -1\end{bmatrix};$$

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5}\end{bmatrix}.$$

We next put $\mathbf{x} = \mathbf{Py}$, to obtain

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{BPy} + \mathbf{P}^{-1}\mathbf{h}(t),$$

that is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2\end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} + \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5}\end{bmatrix}\begin{bmatrix} e^{2t} \\ 4e^{2t}\end{bmatrix}$$

or

$$\dot{y}_1 = 4y_1 + e^{2t}$$
$$\dot{y}_2 = -y_2 - 2e^{2t}.$$

Each of these equations can be solved using the integrating factor method. Writing the first equation in the form

$$\dot{y}_1 - 4y_1 = e^{2t},$$

we see that the integrating factor is e^{-4t}. Multiplying both sides by this gives

$$\frac{d}{dt}(e^{-4t}y_1) = e^{-4t}e^{2t} = e^{-2t},$$

so $\qquad e^{-4t}y_1 = \int e^{-2t}dt = -\frac{1}{2}e^{-2t} + C_1,$

i.e. $\qquad y_1 = C_1e^{4t} - \frac{1}{2}e^{2t}.$

Writing the second equation in the form

$$\dot{y}_2 + y_2 = -2e^{2t},$$

we see that the integrating factor is e^t. Multiplying both sides by this gives

$$\frac{d}{dt}(e^ty_2) = -2e^{3t},$$

so

$$e^ty_2 = -\frac{2}{3}e^{3t} + C_2,$$

i.e.

$$y_2 = C_2e^{-t} - \frac{2}{3}e^{2t}.$$

The solution of the original system is $\mathbf{x} = \mathbf{Py}$; that is,

$$\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -1\end{bmatrix}\begin{bmatrix} C_1e^{4t} - \frac{1}{2}e^{2t} \\ C_2e^{-t} - \frac{2}{3}e^{2t}\end{bmatrix},$$

or

$$x_1 = 3C_1e^{4t} + C_2e^{-t} - \frac{13}{6}e^{2t}$$
$$x_2 = 2C_1e^{4t} - C_2e^{-t} - \frac{1}{3}e^{2t}.$$

2. We have

$$\dot{\mathbf{x}} = \mathbf{Bx} + \mathbf{h}(t)$$

where $\mathbf{B} = \begin{bmatrix} -2 & 10 \\ 10 & -2\end{bmatrix}$ and $\mathbf{h}(t) = [48 \quad 91e^t]^T$.

Solving $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$ gives the eigenvalues of $\mathbf{B}$. Now

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} -2-\lambda & 10 \\ 10 & -2-\lambda\end{vmatrix}$$
$$= (2+\lambda)^2 - 10^2$$
$$= (\lambda + 12)(\lambda - 8).$$

So the eigenvalues are -12 and 8.

Eigenvectors corresponding to the eigenvalue 8 satisfy

$$\begin{bmatrix} -10 & 10 \\ 10 & -10\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix},$$

that is $u = v$. One such eigenvector is $[1 \quad 1]^T$.

Eigenvectors corresponding to the eigenvalue -12 satisfy

$$\begin{bmatrix} 10 & 10 \\ 10 & 10\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix},$$

that is $u + v = 0$. One such eigenvector is $[1 \quad -1]^T$.

Let $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}$. Then

$$\mathbf{P}^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1\end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix},$$

and

$$\mathbf{P}^{-1}\mathbf{BP} = \begin{bmatrix} 8 & 0 \\ 0 & -12\end{bmatrix}.$$

Putting $\mathbf{x} = \mathbf{Py}$ in the original system gives $\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{BPy} + \mathbf{P}^{-1}\mathbf{h}(t)$. That is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2\end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -12\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix} 48 \\ 91e^t\end{bmatrix},$$

or

$$\dot{y}_1 = 8y_1 + 24 + \frac{91}{2}e^t$$
$$\dot{y}_2 = -12y_2 + 24 - \frac{91}{2}e^t.$$

We can solve these equations by the integrating factor method. We obtain (omitting the details)

$$y_1 = C_1e^{8t} - (3 + \frac{13}{2}e^t)$$
$$y_2 = C_2e^{-12t} + (2 - \frac{7}{2}e^t).$$

Then

$$\mathbf{x} = \mathbf{Py} = \begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix} C_1e^{8t} - (3 + \frac{13}{2}e^t) \\ C_2e^{-12t} + (2 - \frac{7}{2}e^t)\end{bmatrix}$$

or

$$x_1 = C_1e^{8t} + C_2e^{-12t} - 1 - 10e^t$$
$$x_2 = C_1e^{8t} - C_2e^{-12t} - 5 - 3e^t.$$

3. (i) As in Example 2, we can write down recurrence relations corresponding to Euler's method directly. We obtain

$$X_{1,r+1} = X_{1,r} + h(3rhX_{2,r} + 4)$$
$$X_{2,r+1} = X_{2,r} + h(rhX_{1,r} - X_{2,r} - e^{rh}).$$

These are already in a form suitable for generating solutions, although it is a little neater to rewrite the second equation:

$$X_{2,r+1} = rh^2X_{1,r} + (1-h)X_{2,r} - he^{rh}.$$

We have also that $X_{1,0} = 5$, $X_{2,0} = 2$.

(ii) With $h = 0.1$ we need to calculate $X_{1,2}$ and $X_{2,2}$ to approximate $x_1(0.2)$ and $x_2(0.2)$. Putting $r = 0$ in the

recurrence, we obtain

$$X_{1,1} = 5 + 0.1 \times 4 = 5.4$$
$$X_{2,1} = 0 + 0.9 \times 2 - 0.1 = 1.7.$$

Putting $r = 1$, we obtain

$$\begin{aligned} X_{1,2} &= X_{1,1} + 0.1(0.3X_{2,1} + 4) \\ &= 5.4 + 0.1 \times 4.51 \\ &= 5.851, \end{aligned}$$

and

$$\begin{aligned} X_{2,2} &= 0.01 \times 5.4 + 0.9 \times 1.7 - 0.1e^{0.1} \\ &= 1.473. \end{aligned}$$

Thus we obtain the approximations $x_1(0.2) \simeq 5.851$ and $x_2(0.2) \simeq 1.473$.

Solutions to the exercises in Section 3

1. To calculate the eigenvalues of $\mathbf{A}$, we first form its characteristic equation:

$$\begin{aligned} 0 &= \begin{vmatrix} -\frac{5}{9} - \lambda & \frac{4}{9} \\ \frac{4}{9} & -\frac{5}{9} - \lambda \end{vmatrix} \\ &= (\lambda + \tfrac{5}{9})^2 - (\tfrac{4}{9})^2 \\ &= (\lambda + 1)(\lambda + \tfrac{1}{9}), \end{aligned}$$

giving the eigenvalues -1, $-\frac{1}{9}$.

To find the corresponding eigenvectors we solve

$$\begin{bmatrix} -\frac{5}{9} - \lambda & \frac{4}{9} \\ \frac{4}{9} & -\frac{5}{9} - \lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

putting λ equal to each eigenvalue in turn.

Case $\lambda = -1$ gives the equations

$$\tfrac{4}{9}u + \tfrac{4}{9}v = 0 \text{ (twice)}.$$

Thus $u = -v$ and so $[1 \quad -1]^T$ is an eigenvector corresponding to the eigenvalue -1.

Case $\lambda = -\frac{1}{9}$ gives the equations

$$-\tfrac{4}{9}u + \tfrac{4}{9}v = 0$$
$$\tfrac{4}{9}u - \tfrac{4}{9}v = 0.$$

Thus $u = v$ and so $[1 \quad 1]^T$ is an eigenvector corresponding to the eigenvalue $-\frac{1}{9}$.

2. The pair of differential equations can be written $\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r}$ where

$$\mathbf{A} = \begin{bmatrix} -\frac{25}{7} & \frac{6}{7} \\ \frac{9}{7} & -\frac{10}{7} \end{bmatrix}.$$

Let us first find the eigenvalues and eigenvectors of $\mathbf{A}$. To find the eigenvalues, form the characteristic equation.

$$\begin{aligned} 0 = |\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} -\frac{25}{7} - \lambda & \frac{6}{7} \\ \frac{9}{7} & -\frac{10}{7} - \lambda \end{vmatrix} \\ &= (\lambda + \tfrac{10}{7})(\lambda + \tfrac{25}{7}) - \tfrac{54}{49} \\ &= \lambda^2 + \tfrac{35}{7}\lambda + \tfrac{196}{49} \\ &= \lambda^2 + 5\lambda + 4 \\ &= (\lambda + 4)(\lambda + 1). \end{aligned}$$

Thus the eigenvalues of $\mathbf{A}$ are -1 and -4.

The eigenvectors corresponding to -1 are given by

$$\begin{bmatrix} -\frac{25}{7} + 1 & \frac{6}{7} \\ \frac{9}{7} & -\frac{10}{7} + 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is

$$-\tfrac{18}{7}u + \tfrac{6}{7}v = 0$$
$$\tfrac{9}{7}u - \tfrac{3}{7}v = 0.$$

Thus $v = 3u$ and so an eigenvector corresponding to the eigenvalue -1 is $[1 \quad 3]^T$.

The eigenvectors for -4 are given by

$$\begin{bmatrix} -\frac{25}{7} + 4 & \frac{6}{7} \\ \frac{9}{7} & -\frac{10}{7} + 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is

$$\tfrac{3}{7}u + \tfrac{6}{7}v = 0$$
$$\tfrac{9}{7}u + \tfrac{18}{7}v = 0.$$

Thus $u = -2v$ and so an eigenvector corresponding to the eigenvalue -4 is $[-2 \quad 1]^T$.

Since the eigenvalues are negative, the object will describe simple harmonic motion along the corresonding eigenvectors. The frequencies will be the square roots of minus the eigenvalues. So there are two possibilities:

(i) motion in the direction of the vector $[1 \quad 3]^T$, with angular frequency 1;

(ii) motion in the direction of the vector $[-2 \quad 1]^T$, with angular frequency 2.

3. (i) We have $\ddot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From the characteristic equation

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1,$$

The eigenvalues of $\mathbf{B}$ are 1 and -1.

The eigenvectors corresponding to the eigenvalue 1 satisfy

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So one such eigenvector is $[1 \quad 1]^T$.

The eigenvectors corresponding to the eigenvalue -1 satisfy

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So one such eigenvector is $[1 \quad -1]^T$.

By Procedure 3.2 the general solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(C_1e^t + D_1e^{-t}) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(C_2\cos t + D_2\sin t)$$

or

$$x_1 = C_1e^t + D_1e^{-t} + C_2\cos t + D_2\sin t$$
$$x_2 = C_1e^t + D_1e^{-t} - C_2\cos t - D_2\sin t.$$

(ii) We must first divide through the second equation by 2, to obtain $\ddot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ with

$$\mathbf{B} = \begin{bmatrix} -10 & -6 \\ -6 & -10 \end{bmatrix}.$$

From the characteristic equation

$$\begin{aligned} 0 &= \begin{vmatrix} -10 - \lambda & -6 \\ -6 & -10 - \lambda \end{vmatrix} \\ &= (\lambda + 10)^2 - 6^2 \\ &= (\lambda + 16)(\lambda + 4) \end{aligned}$$

the eigenvalues of $\mathbf{B}$ are -4 and -16.

The eigenvectors corresponding to the eigenvalue -4 satisfy

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So one such eigenvector is $[1 \quad -1]^T$.

The eigenvectors corresponding to the eigenvalue -16 satisfy

$$\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So one such eigenvector is $[1 \quad 1]^T$.

By Procedure 3.2 the general solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}(C_1 \cos 2t + D_1 \sin 2t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(C_2 \cos 4t + D_2 \sin 4t),$$

or

$$x_1 = C_1 \cos 2t + D_1 \sin 2t + C_2 \cos 4t + D_2 \sin 4t$$
$$x_2 = -C_1 \cos 2t - D_1 \sin 2t + C_2 \cos 4t + D_2 \sin 4t.$$

Solutions to the exercises in Section 4

1. In each case we must write the equation in the form $\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{0}$, and see whether the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ satisfy the specified criteria.

(i) Here

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},$$
$$\mathbf{A}_3 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Each of these matrices is symmetric. $\mathbf{A}_1$ has eigenvalues 1 and 3. We must find the eigenvalues of $\mathbf{A}_2$ and $\mathbf{A}_3$.

For $\mathbf{A}_2$ we have

$$\det(\mathbf{A}_2 - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 2^2 = (\lambda - 1)(\lambda - 5).$$

So $\mathbf{A}_2$ has eigenvalues 1 and 5.

For $\mathbf{A}_3$ we have

$$\det(\mathbf{A}_3 - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - (-1)^2 = (\lambda - 1)(\lambda - 3).$$

So $\mathbf{A}_3$ has eigenvalues 1 and 3.

Hence all the eigenvalues of each matrix are positive.

So the conditions of Theorem 2 *are* satisfied in this case.

(ii) Here

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},$$
$$\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Each of these matrices is symmetric, and $\mathbf{A}_1$ and $\mathbf{A}_2$ have positive eigenvalues. However $\mathbf{A}_3$ has 0 as an eigenvalue, so the conditions of Theorem 2 are *not* satisfied in this case.

[In fact, one can see that $x_1 = 0$, $x_2 = a$ (where a is a non-zero constant) is a solution of this system—and is not a transient.]

(iii) To compare with the form in Theorem 2 we need to bring all the terms to the left-hand side of the equation . So here

$$\mathbf{A}_1 = \mathbf{I}, \quad \mathbf{A}_2 = \mathbf{I}, \quad \mathbf{A}_3 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}.$$

Each of these matrices is symmetric, and $\mathbf{A}_1$ and $\mathbf{A}_2$ have positive eigenvalues. However,

$$\det(\mathbf{A}_3 - \lambda\mathbf{I}) = (-2-\lambda)^2 - (-1)^2 = (-1-\lambda)(-3-\lambda).$$

Hence the eigenvalues of $\mathbf{A}_3$ are negative (-1 and -3), and the conditions of Theorem 2 are not satisfied in this case.

2. If $x_1 = -1$, $x_2 = -t$, then $\ddot{x}_1 = \ddot{x}_2 = 0$, and the equations are satisfied.

From Solution 3(i) in Section 3, the general solution of

$$\ddot{x}_1 = x_2$$
$$\ddot{x}_2 = x_1$$

—the associated homogeneous system—is

$$x_1 = C_1e^t + D_1e^{-t} + C_2 \cos t + D_2 \sin t$$
$$x_2 = C_1e^t + D_1e^{-t} - C_2 \cos t - D_2 \sin t.$$

So the general solution of the given system is

$$x_1 = C_1e^t + D_1e^{-t} + C_2 \cos t + D_2 \sin t - 1$$
$$x_2 = C_1e^t + D_1e^{-t} - C_2 \cos t - D_2 \sin t - t.$$

3. (i) We have a solution of the form $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{2it})$ if

$$-4z_1 + 4z_1 + 2z_2 = 6$$
$$-4z_2 + z_1 + 9z_2 = -2i.$$

That is, if

$$2z_2 = 6$$
$$z_1 + 5z_2 = -2i.$$

Thus $z_2 = 3$, $z_1 = -2i - 15$ and so the required particular solution is

$$x_1 = 2 \sin 2t - 15 \cos 2t,$$
$$x_2 = 3 \cos 2t.$$

(ii) We have a solution $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{it})$ if

$$-z_1 - 2z_2 + iz_1 + z_1 - 3z_2 = -i$$
$$-3z_1 - z_2 + 2iz_2 + 2z_1 + z_2 = 1 + 2i.$$

That is, if

$$iz_1 - 5z_2 = -i$$
$$-z_1 + 2iz_2 = 1 + 2i.$$

Adding i times the second equation to the first gives $-7z_2 = -2$, so

$$z_2 = \tfrac{2}{7}.$$

From the second equation

$$z_1 = \frac{4i}{7} - (1 + 2i) = -1 - \frac{10i}{7}.$$

The required particular solution is therefore

$$x_1 = -\cos t + \tfrac{10}{7} \sin t,$$
$$x_2 = \tfrac{2}{7} \cos t.$$

Solutions to the exercises in Section 5

1.(i) The system can be written $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ with

$$\mathbf{B} = \begin{bmatrix} 8 & -5 \\ 10 & -7 \end{bmatrix}.$$

To find the eigenvalues of $\mathbf{B}$ we solve the characteristic

equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. We have

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} 8-\lambda & -5 \\ 10 & -7-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2),$$

so the eigenvalues of $\mathbf{B}$ are 3 and -2.

The eigenvectors corresponding to the eigenvalue 3 satisfy

$$\begin{bmatrix} 5 & -5 \\ 10 & -10 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $u = v$ and so an eigenvector is $[1 \quad 1]^T$.

The eigenvectors corresponding to the eigenvalue -2 satisfy

$$\begin{bmatrix} 10 & -5 \\ 10 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $v = 2u$ and so an eigenvector is $[1 \quad 2]^T$.

By Procedure 1.2 the general solution of $\dot{\mathbf{x}} = \mathbf{Bx}$ is

$$\mathbf{x} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-2t}.$$

That is

$$x_1 = C_1e^{3t} + C_2e^{-2t}$$
$$x_2 = C_1e^{3t} + 2C_2e^{-2t}.$$

(ii) This can be written $\dot{\mathbf{x}} = \mathbf{Bx} + \mathbf{h}(t)$ where $\mathbf{B}$ is the same as in (i) and $\mathbf{h}(t) = [2e^t \quad -e^{2t}]^T$. The general solution can be found using Procedure 2.1.

The eigenvalues and eigenvectors of $\mathbf{B}$ are given in (i).

We form the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and put $\mathbf{x} = \mathbf{Py}$. This gives

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{BPy} + \mathbf{P}^{-1}\begin{bmatrix} 2e^t \\ -e^{2t} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2e^t \\ -e^{2t} \end{bmatrix}.$$

That is

$$\dot{y}_1 = 3y_1 + 4e^t + e^{2t}$$
$$\dot{y}_2 = -2y_2 - 2e^t - e^{2t}.$$

Using the integrating factor method we find that the general solutions of these equations are

$$y_1 = C_1e^{3t} - 2e^t - e^{2t}$$
$$y_2 = C_2e^{-2t} - \tfrac{2}{3}e^t - \tfrac{1}{4}e^{2t}.$$

So the general solution of the original system is $\mathbf{x} = \mathbf{Py}$, that is

$$x_1 = C_1e^{3t} + C_2e^{-2t} - \tfrac{8}{3}e^t - \tfrac{5}{4}e^{2t}$$
$$x_2 = C_1e^{3t} + 2C_2e^{-2t} - \tfrac{10}{3}e^t - \tfrac{3}{2}e^{2t}.$$

(iii) This can be written $\dot{\mathbf{x}} = \mathbf{Bx}$ where

$$\mathbf{B} = \begin{bmatrix} 6 & -4 \\ 10 & -6 \end{bmatrix}.$$

The eigenvalues and eigenvectors of this matrix are complex. One eigenvalue is $\lambda = 2i$, with corresponding eigenvector $\mathbf{a} = [4 \quad 6 - 2i]^T$.

Now

$$\mathbf{a}e^{\lambda t} = \begin{bmatrix} 4 \\ 6-2i \end{bmatrix}e^{2it} = \begin{bmatrix} 4(\cos 2t + i\sin 2t) \\ (6-2i)(\cos 2t + i\sin 2t) \end{bmatrix},$$

so

$$\mathrm{Re}(\mathbf{a}e^{\lambda t}) = \begin{bmatrix} 4\cos 2t \\ 6\cos 2t + 2\sin 2t \end{bmatrix},$$
$$\mathrm{Im}(\mathbf{a}e^{\lambda t}) = \begin{bmatrix} 4\sin 2t \\ 6\sin 2t - 2\cos 2t \end{bmatrix}.$$

By Procedure 1.2(a) the general solution is

$$\mathbf{x} = C_1\mathrm{Re}(\mathbf{a}e^{\lambda t}) + C_2\mathrm{Im}(\mathbf{a}e^{\lambda t}),$$

or

$$x_1 = 4C_1\cos 2t + 4C_2\sin 2t$$
$$x_2 = C_1(6\cos 2t + 2\sin 2t) + C_2(6\sin 2t - 2\cos 2t).$$

(iv) In matrix form this can be written $\ddot{\mathbf{x}} = \mathbf{Bx}$ where $\mathbf{B}$ is the same matrix as in (i). The eigenvalues and eigenvectors of $\mathbf{B}$ are given in (i). By Procedure 3.2 the general solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(C_1e^{\sqrt{3}t} + D_1e^{-\sqrt{3}t}) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}(C_2\sin\sqrt{2}t + D_2\cos\sqrt{2}t)$$

or

$$x_1 = C_1e^{\sqrt{3}t} + D_1e^{-\sqrt{3}t} + C_2\sin\sqrt{2}t + D_2\cos\sqrt{2}t.$$
$$x_2 = C_1e^{\sqrt{3}t} + D_1e^{-\sqrt{3}t} + 2C_2\sin\sqrt{2}t + 2D_2\cos\sqrt{2}t.$$

(v) The general solution is the sum of the complementary function, which we found in solution (iv), and any one particular solution. We can find a sinusoidal particular solution by applying Procedure 4.2.

Putting $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{2it})$ leads to the equations

$$-4z_1 = 8z_1 - 5z_2 - i$$
$$-4z_2 = 10z_1 - 7z_2 + 2,$$

which have the solution

$$z_1 = -\tfrac{5}{7} - \tfrac{3}{14}i, \qquad z_2 = -\tfrac{12}{7} - \tfrac{5}{7}i.$$

Hence a particular solution of the system is

$$x_1 = -\tfrac{5}{7}\cos 2t + \tfrac{3}{14}\sin 2t$$
$$x_2 = -\tfrac{12}{7}\cos 2t + \tfrac{5}{7}\sin 2t.$$

The general solution of the system is therefore

$$x_1 = C_1e^{\sqrt{3}t} + D_1e^{-\sqrt{3}t} + C_2\sin\sqrt{2}t + D_2\cos\sqrt{2}t - \tfrac{5}{7}\cos 2t + \tfrac{3}{14}\sin 2t$$
$$x_2 = C_1e^{\sqrt{3}t} + D_1e^{-\sqrt{3}t} + 2C_2\sin\sqrt{2}t + 2D_2\cos\sqrt{2}t - \tfrac{12}{7}\cos 2t + \tfrac{5}{7}\sin 2t.$$

(vi) This equation can be written $\dot{\mathbf{x}} = \mathbf{Bx}$ where

$$\mathbf{B} = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix}.$$

To find the eigenvalues of $\mathbf{B}$ we solve the characteristic equation $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. Now

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 2 & 0 \\ -1 & -3-\lambda & 0 \\ 0 & 1 & -4-\lambda \end{vmatrix}$$
$$= (-4-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{vmatrix}$$
$$= (-4-\lambda)(\lambda^2 + 4\lambda + 5).$$

So the eigenvalues of $\mathbf{B}$ are -4 and $-2 \pm i$.

The eigenvectors corresponding to the eigenvalue -4 satisfy

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So $u = v = 0$, and an eigenvector is $[0 \quad 0 \quad 1]^T$.

The eigenvectors corresponding to the eigenvalue $-2+i$ satisfy

$$\begin{bmatrix} 1-i & 2 & 0 \\ -1 & -1-i & 0 \\ 0 & 1 & -2-i \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is

$$(1-i)u + 2v = 0$$
$$-u - (1+i)v = 0$$
$$v - (2+i)w = 0.$$

The first equation is $(i-1)$ times the second equation. The second equation gives $u = -(1+i)v$ and the third equation gives $w = v/(2+i) = \frac{1}{5}(2-i)v$. Thus an eigenvector corresponding to the eigenvalue $\lambda = -2+i$ is
$\mathbf{a} = [-(1+i) \quad 1 \quad \frac{1}{5}(2-i)]^T$.

The general solution of the given system can be found using Procedure 1.2(a). We have

$$\mathbf{a}e^{\lambda t} = e^{-2t}\begin{bmatrix} -1-i \\ 1 \\ \frac{1}{5}(2-i) \end{bmatrix}(\cos t + i\sin t)$$

$$= e^{-2t}\left(\begin{bmatrix} -\cos t + \sin t \\ \cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{bmatrix} + i\begin{bmatrix} -\sin t - \cos t \\ \sin t \\ \frac{1}{5}(2\sin t - \cos t) \end{bmatrix}\right),$$

so

$$\mathrm{Re}(\mathbf{a}e^{\lambda t}) = e^{-2t}\begin{bmatrix} -\cos t + \sin t \\ \cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{bmatrix},$$

$$\mathrm{Im}(\mathbf{a}e^{\lambda t}) = e^{-2t}\begin{bmatrix} -\sin t - \cos t \\ \sin t \\ \frac{1}{5}(2\sin t - \cos t) \end{bmatrix}.$$

The general solution is therefore

$$\mathbf{x} = C_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}e^{-4t} + C_2\begin{bmatrix} -\cos t + \sin t \\ \cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{bmatrix}e^{-2t}$$
$$+ C_3\begin{bmatrix} -\sin t - \cos t \\ \sin t \\ \frac{1}{5}(2\sin t - \cos t) \end{bmatrix}e^{-2t},$$

or

$$x_1 = C_2e^{-2t}(-\cos t + \sin t) + C_3e^{-2t}(-\sin t - \cos t)$$
$$x_2 = C_2e^{-2t}\cos t + C_3e^{-2t}\sin t$$
$$x_3 = C_1e^{-4t} + \tfrac{1}{5}C_2e^{-2t}(2\cos t + \sin t) + \tfrac{1}{5}C_3e^{-2t}(2\sin t - \cos t).$$

(vii) We first write the equations in standard form. To do this, we separate the derivative and non-derivative terms:

$$\dot{x}_1 + \dot{x}_2 = 4x_1 + 4x_2$$
$$\dot{x}_2 = 3x_1 + x_2 + x_3$$
$$\dot{x}_3 = 4x_3.$$

and then write this in the matrix form $\mathbf{A}\dot{\mathbf{x}} = \mathbf{C}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 4 & 4 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

The normal form of the equation is

$$\dot{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{C}\mathbf{x}.$$

The inverse $\mathbf{A}^{-1}$ can be found using row operations (see *Unit 20*, Subsection 4.2). We have

$$\begin{array}{c} \\ \\ \\ \\ \mathbf{R}_1 - \mathbf{R}_2 \\ \\ \\ \\ \end{array}\begin{array}{c} \qquad\mathbf{A} \qquad\qquad\qquad \mathbf{I} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ \qquad\mathbf{I} \qquad\qquad\qquad \mathbf{A}^{-1} \end{array}\begin{array}{c} \\ \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \\ \\ \\ \\ \end{array}$$

Thus

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So the normal form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}\mathbf{x}$$

$$= \begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}\mathbf{x}.$$

We next need the eigenvalues and eigenvectors of

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

These were calculated in Example 1 of Section 3 where we found

eigenvalues	corresponding eigenvectors
-2	$[1 \quad -1 \quad 0]^T$
4	$[1 \quad 1 \quad 0]^T$ and $[0 \quad 1 \quad 3]^T$

By Procedure 1.2 the general solution is

$$\mathbf{x} = C_1e^{-2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + C_2e^{4t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3e^{4t}\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix},$$

or

$$x_1 = C_1e^{-2t} + C_2e^{4t}$$
$$x_2 = -C_1e^{-2t} + C_2e^{4t} + C_3e^{4t}$$
$$x_3 = 3C_3e^{4t}.$$

2. The system can be written in the form

$$\mathbf{A}_1\ddot{\mathbf{x}} + \mathbf{A}_2\dot{\mathbf{x}} + \mathbf{A}_3\mathbf{x} = \mathbf{h}(t)$$

with

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix},$$

$$\mathbf{h}(t) = \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.$$

Each of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ is symmetric. The eigenvalues of $\mathbf{A}_1$ are 3 and 5 for

$(\det(\mathbf{A}_1 - \lambda\mathbf{I}) = (4-\lambda)^2 - 1^2 = (\lambda - 3)(\lambda - 5)$,

and those of $\mathbf{A}_2$ are 3 and 3 — all positive. Those of $\mathbf{A}_3$ are a little harder to find. We have

$$\begin{aligned}\det(\mathbf{A}_3 - \lambda\mathbf{I}) &= (3-\lambda)(6-\lambda) - 16 \\ &= \lambda^2 - 9\lambda + 2.\end{aligned}$$

So the eigenvalues of $\mathbf{A}_3$ are $\frac{1}{2}(9 \pm \sqrt{9^2 - 8})$. Written in this form we can see that these eigenvalues are positive, since $\sqrt{9^2 - 8} < 9$.

Hence we may deduce from Theorem 2 of Section 4 that the complementary function of the given system is a transient.

There is a unique sinusoidal particular solution of the given system. Every solution of the given system converges to this same solution as t becomes large. Let us now calculate this solution.

To do this we put $\mathbf{x} = \mathrm{Re}(\mathbf{z}e^{2it})$. This yields the equations

$$\begin{aligned}-16z_1 - 4z_2 + 6iz_1 + 3z_1 + 4z_2 &= 1 \\ -4z_1 - 16z_2 + 6iz_2 + 4z_1 + 6z_2 &= -i.\end{aligned}$$

That is

$$\begin{aligned}(6i - 13)z_1 &= 1 \\ (6i - 10)z_2 &= -i.\end{aligned}$$

So

$$z_1 = \frac{1}{6i - 13} = -\frac{6i + 13}{205},$$

$$z_2 = \frac{i(6i + 10)}{136} = \frac{-6 + 10i}{136}.$$

The steady-state solution is therefore

$$\begin{aligned}x_1 &= -\tfrac{1}{205}(13\cos 2t - 6\sin 2t) \\ x_2 &= -\tfrac{1}{136}(6\cos 2t + 10\sin 2t).\end{aligned}$$

3. The relevant recurrence relations, for step-length h, may be written down directly:

$$\begin{aligned}X_{1,r+1} &= X_{1,r} + hX_{2,r} \\ X_{2,r+1} &= X_{2,r} + hX_{3,r} \\ X_{3,r+1} &= X_{3,r} + h(X_{1,r} + 3X_{2,r} - rhX_{3,r} - \cos rh)\end{aligned}$$

where $X_{1,0} = 2$, $X_{2,0} = -1$, $X_{3,0} = 4$.

(Here $X_r = [X_{1,r} \quad X_{2,r} \quad X_{3,r}]^T$ approximates the true value of $\mathbf{x}(rh)$.)

Appendix 2: Solutions to the problems

Solutions to the problems in Section 5

1. **(i)** We have $y = x_1$, $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, and

$$\dot{x}_3 = x_1 + 3x_2 - tx_3 - \cos t. \qquad (1)$$

We can express the $\dot{x}_3$, x_1, x_2 and x_3 in (1) in terms of y as follows

$$x_1 = y,$$
$$x_2 = \dot{x}_1 = \dot{y},$$
$$x_3 = \dot{x}_2 = \ddot{y},$$
$$\dot{x}_3 = \dddot{y}.$$

Substituting these into Equation (1) gives

$$\dddot{y} = y + 3\dot{y} - t\ddot{y} - \cos t,$$

or

$$\dddot{y} + t\ddot{y} - 3\dot{y} - y = -\cos t,$$

as required.

(ii) We are given that y satisfies $\dddot{y} + 4ty = \sin 2t$. To reverse the process in (i), we introduce some new variables. We define x_1, x_2 and x_3 by the equations

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}.$$

Then we have

$$\dot{x}_1 = x_2$$

and

$$\dot{x}_2 = x_3.$$

Also $\dot{x}_3 = \dddot{y}$, so from the given differential equation,

$$\dot{x}_3 = \sin 2t - 4tx_1.$$

Thus the given equation is equivalent to the first-order system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \sin 2t - 4tx_1.$$

The initial conditions give

$$x_1(0) = 1, \qquad x_2(0) = 2, \qquad x_3(0) = 0.$$

We can use the method described in Subsection 2.2 to write down recurrence relations to generate approximate solutions to this system:

$$X_{1,r+1} = X_{1,r} + hX_{2,r}$$
$$X_{2,r+1} = X_{2,r} + hX_{3,r}$$
$$X_{3,r+1} = X_{3,r} + h(\sin 2rh - 4rhX_{1,r}).$$

with $\quad X_{1,0} = 1$, $X_{2,0} = 2$, $X_{3,0} = 0$.

If we use these recurrence relations, then $X_{1,r}$ will provide an approximation to the solution $y(rh)$ of the original differential equation.

(The method described in this problem generalises the method of numerical approximation for a single *second*-order differential equation described in *Unit 6*.)

(iii) If we put $x_1 = x$, $\quad x_2 = y$, $\quad x_3 = \dot{x}$, $\quad x_4 = \dot{y}$, we have

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 (= \ddot{x}) = \frac{-kx_1}{x_1^2 + x_2^2} - \lambda(x_3^2 + x_4^2)^{1/2}x_3$$
$$\dot{x}_4 (= \ddot{y}) = \frac{-kx_2}{x_1^2 + x_2^2} - \lambda(x_3^2 + x_4^2)^{1/2}x_4.$$

The following recurrence relations are obtained if we apply Euler's method to each equation in turn (with step length h).

$$X_{1,r+1} = X_{1,r} + hX_{3,r}$$
$$X_{2,r+1} = X_{2,r} + hX_{4,r}$$
$$X_{3,r+1} = X_{3,r} - h\left[\frac{kX_{1,r}}{X_{1,r}^2 + X_{2,r}^2} + \lambda(X_{3,r}^2 + X_{4,r}^2)^{1/2}X_{3,r}\right]$$
$$X_{4,r+1} = X_{4,r} - h\left[\frac{kX_{2,r}}{X_{1,r}^2 + X_{2,r}^2} + \lambda(X_{3,r}^2 + X_{4,r}^2)^{1/2}X_{4,r}\right].$$

2. **(i)** If

$$\mathbf{x} = \mathbf{a}e^{\lambda t} + \mathbf{b}te^{\lambda t} \qquad (1)$$

then

$$\dot{\mathbf{x}} = \lambda\mathbf{a}e^{\lambda t} + \mathbf{b}(e^{\lambda t} + \lambda te^{\lambda t}).$$

So (1) is a solution of $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ so long as

$$(\lambda\mathbf{a} + \mathbf{b})e^{\lambda t} + \mathbf{b}\lambda te^{\lambda t} = \mathbf{B}(\mathbf{a}e^{\lambda t} + \mathbf{b}te^{\lambda t}).$$

This is the case so long as

$$\lambda\mathbf{a} + \mathbf{b} = \mathbf{B}\mathbf{a} \text{ (equating coefficients of } e^{\lambda t}), \qquad (2)$$

and $\quad \lambda\mathbf{b} = \mathbf{B}\mathbf{b} \quad$ (equating coefficients of $te^{\lambda t}$). $\qquad (3)$

That is, (a) $\mathbf{b}$ is an eigenvector of $\mathbf{B}$, with eigenvalue λ (from (3)), and (b) $\mathbf{a}$ satisfies $(\mathbf{B} - \lambda\mathbf{I})\mathbf{a} = \mathbf{b}$ (from (2)), as required.

(ii) **(a)** We have $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ with

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

The eigenvalues satisfy

$$0 = \begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Thus 2 is the only eigenvalue. The corresponding eigenvectors satisfy

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $v = 2u$, and so every eigenvector is of the form $[k \quad 2k]^T = k[1 \quad 2]^T$.

To be able to apply Procedure 1.2 of Section 1 we require two linearly independent eigenvectors, but this is impossible since all eigenvectors are multiples of $[1 \quad 2]^T$.

(b) Since $[1 \quad 2]^T$ is an eigenvector corresponding to the eigenvalue 2 we know that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{2t}$$

is one solution of $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$. We can use part (i) of the problem to find a second linearly independent solution. To do this we need to find a vector $\mathbf{a}$ satisfying $(\mathbf{B} - \lambda\mathbf{I})\mathbf{a} = \mathbf{b}$, where $\mathbf{b}$ is the eigenvector $[1 \quad 2]^T$. That is, if $\mathbf{a} = [a_1 \quad a_2]^T$ we need

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

That is

$$-2a_1 + a_2 = 1$$
$$-4a_1 + 2a_2 = 2.$$

These two equations are equivalent, so there is a wide choice of solutions. One is $a_1 = 0$, $\quad a_2 = 1$ giving $\mathbf{a} = [0 \quad 1]^T$.

With these values for **a** and **b**, we know from part (i) that

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} te^{2t}$$

is a solution of $\dot{\mathbf{x}} = \mathbf{Bx}$.

(c) The general solution is

$$\mathbf{x} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} te^{2t} \right)$$

or

$$x_1 = C_1 e^{2t} + C_2 te^{2t}$$
$$x_2 = 2C_1 e^{2t} + C_2 e^{2t} + 2C_2 te^{2t}.$$

(To get this we assume the fact that the general solution is a linear combination of the two solutions we have found.)

(iii) We now have $\dot{\mathbf{x}} = \mathbf{Ax}$, with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

The eigenvalues of **A** satisfy

$$0 = \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1).$$

Thus the eigenvalues are 2 and 1.

The eigenvectors corresponding to the eigenvalue 1 satisfy

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $u = v$, and so an eigenvector is $[1 \quad 1]^T$.

Let us try to find a solution of the form specified in part (i). We would need a vector $\mathbf{a} = [a_1 \quad a_2]^T$ satisfying

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

That is

$$a_2 - a_1 = 1$$
$$2a_2 - 2a_1 = 1.$$

These equations are inconsistent and have no solutions.

If we proceed similarly for the eigenvalue 2 we again arrive at a set of inconsistent equations for **a**. So in this case *no* solution of the form specified in part (i) can be found. This is just as well, as we have enough independent solutions already!

(iv) **(a)** In part (ii) we had $\mathbf{b} = [1 \quad 2]^T$, $\mathbf{a} = [0 \quad 1]^T$. [You may have different vectors here—there was more than one valid choice of these vectors.] Thus

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Then

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},$$

and so

$$\mathbf{P}^{-1}\mathbf{BP} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

(b) The given system can be written

$$\dot{\mathbf{x}} = \mathbf{Bx} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

Putting $\mathbf{x} = \mathbf{Py}$ gives

$$\mathbf{P}\dot{\mathbf{y}} = \mathbf{BPy} + \begin{bmatrix} 0 \\ e^t \end{bmatrix},$$

so

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{BPy} + \mathbf{P}^{-1} \begin{bmatrix} 0 \\ e^t \end{bmatrix},$$

i.e.

$$\dot{\mathbf{y}} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

Thus

$$\dot{y}_1 = 2y_1 + y_2$$
$$\dot{y}_2 = 2y_2 + e^t.$$

We can solve the second equation, by the integrating factor method to obtain

$$y_2 = C_1 e^{2t} - e^t.$$

Substituting this into the first equation gives

$$\dot{y}_1 = 2y_1 + C_1 e^{2t} - e^t.$$

We can now solve this by the integrating factor method, to obtain

$$y_1 = C_2 e^{2t} + C_1 te^{2t} + e^t.$$

(c) Now $\mathbf{x} = \mathbf{Py}$, so we have

$$\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_2 e^{2t} + C_1 te^{2t} + e^t \\ C_1 e^{2t} - e^t \end{bmatrix}.$$

Thus

$$x_1 = C_2 e^{2t} + C_1 te^{2t} + e^t$$
$$x_2 = 2C_2 e^{2t} + C_1 e^{2t}(1 + 2t) + e^t$$

is the general solution of the given system.

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